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The R-map and the coupling of $\mathcal{N} = 2$ tensor multiplets in 5 and 4 dimensions

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Abstract: We study the dimensional reduction of 5D, $\mathcal{N} = 2$ Yang-Mills-Einstein supergravity theories (YMESGT) coupled to tensor multiplets. The resulting 4D theories involve first order interactions among tensor and vector fields with mass terms. If the 5D gauge group, K, does not mix the 5D tensor and vector fields, the 4D tensor fields can be integrated out in favor of the 4D vector fields and the resulting theory is dual to a standard 4D YMESGT (Integrating out the vector fields in favor of tensor fields instead seems to require nonlocal field redefinitions). The gauge group has a block diagonal symplectic embedding and is a semi-direct product of the 5D gauge group K with a Heisenberg group $\mathcal{H}^{n_T+1}$ of dimension $n_T + 1$, where n_T is the number of tensor fields in five dimensions. There exists an infinite family of theories, thus obtained, whose gauge groups are pp-wave contractions of the simple noncompact groups of type $\mathrm{SO}^*(2N)$. If, on the other hand, the 5D gauge group does mix the 5D tensor and vector fields, the resulting 4D theory is dual to a 4D YMESGT whose gauge group does, in general, *not* have a block diagonal symplectic embedding and involves additional topological terms. The scalar potentials of the dimensionally reduced theories studied in this paper naturally have some of the ingredients that were found necessary for stable de Sitter ground states in earlier studies. We comment on the relation between the known 5D and 4D, $\mathcal{N} = 2$ supergravities with stable de Sitter ground states.

Keywords: Supergravity Models, Supersymmetry and Duality, Extended Supersymmetry.

Contents

1. Introduction

Four-dimensional supergravity theories with massive antisymmetric tensor fields[1] have recently received a lot of attention [2−4] due to their relevance for string compactifications with background fluxes [5] or Scherk-Schwarz generalized dimensional reduction [6].

[1]For some earlier work, see also [1].

In conventional string compactifications *without* background fluxes or geometric twists, *massless* two-forms in the effective 4D theory naturally descend from the various types of p-form fields in the 10D or 11D actions of string or M-theory. A massless two-form in 4D is Hodge dual to a massless scalar field, and upon such a dualization, the 4D effective theory is readily expressed in terms of scalar fields and vector fields only (plus the gravitational sector and the fermions). In an $\mathcal{N} = 2$ compactification, the resulting theories then describe the coupling of massless vector and hypermultiplets to supergravity without gauge interactions.

When fluxes or geometric twists are switched on, however, the low energy effective theories typically contain gauge interactions and mass deformations, which in turn entail non-trivial scalar potentials.[2] In the presence of mass deformations for two-form fields, the massive two-form can no longer be directly dualized to a scalar field. Instead, a massive two-form is dual to a massive vector field [9], and the relation to the standard formulation of 4D gauged supergravity in terms of scalar fields and vector fields [10, 11] is, a priori, less clear. In the well understood cases, this relation involves the gauging of axionic isometries on the scalar manifold, upon which the axionic scalar field can be "eaten" by a vector field to render it massive [2 − 4].

In the context of 4D, $\mathcal{N} = 2$ supergravity, such mass deformations have been primarily studied for two-forms that, before the deformation, arise from dualizations of scalars of the quaternionic Kähler manifold of the hypermultiplet sector [2]. It was only very recently that such mass deformations were also studied for tensor fields that are dual to scalars of the special Kähler manifold [12].

Massive tensor fields also play an important rôle in 5D, $\mathcal{N} = 2$ gauged supergravity [13]. In five dimensions, massless tensor fields are dual to massless vector fields when they are not charged with respect to any local gauge symmetry. In ungauged supergravity, two-form fields are therefore usually replaced by vector fields [14]. When gauge interactions are turned on, however, the equivalence between two-form fields and vector fields is typically lost, and one has to distinguish between them more carefully [15 − 18, 13, 19]. In particular, two-form fields that transform non-trivially under some gauge group are possible, and such two-forms are no longer equivalent to vector fields.[3] This can be understood from the fact that the charged tensor fields acquire a mass, and massive tensors in 5D have a different number of degrees of freedom than vectors. In the conventional formulation of such 5D gauged supergravity theories with tensor fields, the tensor fields $B^M_{\mu\nu}$ enter the Lagrangian via first order terms of the form [16 − 18, 13, 19]

$$\Omega_{MN} B^M \wedge \mathcal{D} B^N \tag{1.1}$$

where Ω_{MN} is a symplectic metric, $\mathcal{D}B^N = dB^N + g\Lambda^N_{IM} A^I \wedge B^M$, and Λ^N_{IM} denotes the transformation matrix of the tensor fields with respect to the gauge group gauged

[2]Gauge interactions and non-trivial scalar potentials can also occur when the compactifying manifold exhibits certain types of singularities or is close to other special points in the moduli space, corresponding, e.g. to self-dual radii of circles etc. These gauge interactions and potentials are often associated with additional light states, which, in the case of singularities, are typically localized at those singularities as e.g. in [7] (for a complete treatment of a concrete example in the language of gauged supergravity see also [8]).

[3]A reformulation of 5D, $\mathcal{N} = 8$ gauged supergravity which treats vector and tensor fields more symmetrically has recently been given in [20].

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by the vector fields A^I_μ and with gauge coupling g. Reiterating the resulting field equations, half of the tensors can be eliminated, and one obtains second order field equations for the remaining ones with mass terms due to a $B^M \wedge *B^N$ coupling in the Lagrangian.

Whereas the dimensional reduction of *ungauged* 5D supergravity to 4D has been studied quite extensively in the literature, surprisingly little is known about the dimensional reduction of 5D *gauged* supergravity *with tensor fields*. For example, the $\mathcal{N} = 8$ AdS graviton supermultiplet involves both vector and tensor fields in five dimensions [21]. Hence gauging the maximal supergravity in five dimensions requires that some of the vector fields of the ungauged theory be dualized to tensor multiplets [16, 17]. Remarkably, the SU(3, 1) gauged $\mathcal{N} = 8$ supergravity constructed in [22] has a stable ground state that preserves two supersymmetries and has a vanishing cosmological constant. The general properties of the compactification of the SU(3, 1) gauged 5D, $\mathcal{N} = 8$ supergravity down to four dimensions were originally investigated in [22]. More recently, a more detailed analysis of the dimensional reduction of 5D gauged $\mathcal{N} = 8$ supergravity down to four dimensions was given by Hull [23], but to the best of our knowledge, a complete analysis, in particular for $\mathcal{N} = 2$, has never been given. As the naive dimensional reduction is expected to involve massive two-forms of some sort, it is important to close this gap in the literature and to compare the result with the current work on 4D massive two-forms [3, 2, 4, 20] and the standard formulation of gauged supergravity theories in 4D [10, 11].

As the resulting theory only involves the (very) special Kähler gemetry of the vector multiplet sector in 4D, and since the tensors are expected to transform nontrivially under (in general non-Abelian) gauge symmetries, the resulting theories are expected to be different from the ones studied in the recent works [2] on hypermultiplet scalars.

The dimensional reduction of 5D, $\mathcal{N} = 2$ gauged supergravity with tensor multiplets to 4D could also be interesting for the recent attempts to find stable de Sitter ground states in extended supergravity theories [24 – 27]. So far, the only known examples for such stable de Sitter vacua were found in 5D, $\mathcal{N} = 2$ gauged supergravity theories with tensor fields [24, 27] and in certain 4D, $\mathcal{N} = 2$ gauged supergravity theories [26]. As for the latter type of theories, the authors of [26] identified a number of ingredients that were necessary to obtain stable de Sitter vacua. These include non-Abelian non-compact gauge groups, de Roo-Wagemans rotation angles [28] and gaugings of subgroups of the R-symmetry group. Interestingly, gaugings of the R-symmetry group also play a rôle for the known 5D theories with stable de Sitter vacua [24, 27]. Also, the known 5D examples involve non-compact gauge groups. However, in 5D, these groups can be Abelian and still give rise to stable de Sitter vacua. Furthermore, the known 5D models involve charged tensor multiplets, whereas de Roo-Wagemans rotation angles are not well-defined in 5D. One of the important results of our paper is that the dimensional reduction of 5D, $\mathcal{N} = 2$ gauged supergravity with tensor multiplets to 4D always leads to non-Abelian non-compact gauge groups, no matter what the 5D gauge group is. Furthermore, one always introduces something similar to de Roo-Wagemans rotation angles in this reduction process. We do not consider gaugings of the R-symmetry group in this paper, but putting the above together, one might wonder whether the dimensionally reduced 5D theories with tensor fields could give rise to 4D stable de Sitter vacua, perhaps after switching on R-symmetry gaugings and/or suitable truncations or extensions.

Motivated by these and other possible applications, we will, in this paper, systematically study the dimensional reduction of 5D, $\mathcal{N}=2$ gauged supergravity with tensor multiplets to 4D.

The outline of this paper is as follows. In section 2, we briefly recapitulate the structure of 5D, $\mathcal{N}=2$ ungauged Maxwell-Einstein supergravity theories (MESGTs). In section 3, we review the gaugings of these theories which require the introduction of tensor fields. Here, two cases are to be distinguished: (i) The vector fields of the ungauged theory transform in a completely reducible representation of the prospective gauge group, or (ii) they form a representation that is reducible, but not *completely* reducible [29]. In section 4, we dimensionally reduce the theories of type (i) to 4D. Section 5 discusses the rôle played by the massive two-forms and vector fields in the resulting 4D theories. The dimensionally reduced theories have a first order interaction between two-form and vector fields that is reminiscent of the Freedman-Townsend model [30] and looks like a concrete realization of the formalism of [12]. We then eliminate the tensor fields in favor of vector fields, which are indeed massive. The opposite elimination of the vector fields in terms of the tensor fields meets some difficulties and might be possible only in a rather non-trivial way. In section 6, we show that, after suitable symplectic rotations, the resulting theory without the two-forms can be mapped to a standard gauged supergravity theory in 4D in which the gauge group has a block diagonal symplectic embedding. This theory has a gauge group of the form $(K \ltimes \mathcal{H}^{n_T+1})$, which is the semidirect product of the 5D gauge group K with the (n_T+1)-dimensional Heisenberg group $\mathcal{H}^{n_T+1}$ generated by n_T translation generators and a central charge (n_T denotes the number of tensor multiplets in five dimensions, which is always even). The case (ii) of not completely recducible representations is briefly sketched in section 6.3. The dimensional reduction of theories with completely reducible representations in 5D parallels the situation in the $\mathcal{N}=8$ theory described by Hull in [23], as explained in section 7, where we also comment on the relation to the "unified" supergravity theories studied in [31]. In section 8, we study some properties of the scalar potential and comment on the relation to extended supergravity theories with stable de Sitter ground states. Appendix A, finally, contains some details of the dimensional reduction.

2. 5D, $\mathcal{N}=2$ Maxwell-Einstein supergravity theories

Five-dimensional minimal supergravity can be coupled to vector, tensor and hypermultiplets [14, 32, 33, 13, 34, 35, 29]. Hypermultiplets are irrelevant for this paper and will henceforth be ignored. In five dimensions, massless uncharged vector fields and massless uncharged two-form fields are dual to one another. At the level of ungauged supergravity theories, the distinction between vector and tensor multiplets is therefore unnecessary, and one can, without loss of generality, dualize all tensor fields to vector fields. These theories are often referred to as "Maxwell-Einstein supergravity theories" ("MESGTs") and were first constructed in [14]. Our notation in this paper follows that of [14, 13], except that we will put a hat on all five-dimensional spacetime and tangent space indices, as well as on all fields that decompose nontrivially into four-dimensional fields, as will become obvious below.

The 5D, $\mathcal{N}=2$ supergravity multiplet consists of the fünfbein $\hat{e}_{\hat{\mu}}^{\hat{m}}$, two gravitini $\hat{\psi}_{\hat{\mu}}^{i}$ $(i=1,2)$ and one vector field $\hat{A}_{\hat{\mu}}$ (the "graviphoton"). A vector multiplet contains a vector field $\hat{A}_{\hat{\mu}}$, two "gaugini" $\hat{\lambda}^i$ and one real scalar field, φ. Coupling $\tilde{n}$ vector multiplets to supergravity, the total bosonic field content is then

$$\{\hat{e}_{\hat{\mu}}^{\hat{m}}, \hat{A}_{\hat{\mu}}^{\tilde{I}}, \varphi^{\tilde{x}}\},$$

where, as usual, the graviphoton and the $\tilde{n}$ vector fields from the $\tilde{n}$ vector multiplets have been combined into one $(\tilde{n}+1)$-plet of vector fields $\hat{A}_{\hat{\mu}}^{\tilde{I}}$ $(\tilde{I}=1,\ldots,\tilde{n}+1)$. The indices $\tilde{x},\tilde{y},\ldots$ denote the curved indices of the $\tilde{n}$-dimensional target manifold, $\mathcal{M}^{(5)}$, of the scalar fields.

The bosonic part of the Lagrangian is given by (for the fermionic part and further details, see [14])

$$\begin{aligned}\mathcal{L}^{(5)} = & -\frac{1}{2}\hat{e}\hat{R} - \frac{1}{4}\hat{e}\overset{\circ}{a}_{\tilde{I}\tilde{J}}\hat{F}_{\hat{\mu}\hat{\nu}}^{\tilde{I}}\hat{F}^{\tilde{J}\hat{\mu}\hat{\nu}} - \frac{\hat{e}}{2}g_{\tilde{x}\tilde{y}}(\partial_{\hat{\mu}}\varphi^{\tilde{x}})(\partial^{\hat{\mu}}\varphi^{\tilde{y}}) \\ & + \frac{1}{6\sqrt{6}}C_{\tilde{I}\tilde{J}\tilde{K}}\hat{\epsilon}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\lambda}}\hat{F}_{\hat{\mu}\hat{\nu}}^{\tilde{I}}\hat{F}_{\hat{\rho}\hat{\sigma}}^{\tilde{J}}\hat{A}_{\hat{\lambda}}^{\tilde{K}}\end{aligned} \tag{2.1}$$

where $\hat{e}$ and $\hat{R}$ denote, respectively, the fünfbein determinant and scalar curvature of spacetime. $\hat{F}_{\hat{\mu}\hat{\nu}}^{\tilde{I}} \equiv 2\partial_{[\hat{\mu}}\hat{A}_{\hat{\nu}]}^{\tilde{I}}$ are the standard Abelian field strengths of the vector fields $\hat{A}_{\hat{\mu}}^{\tilde{I}}$. The metric, $g_{\tilde{x}\tilde{y}}$, of the scalar manifold $\mathcal{M}^{(5)}$ and the matrix $\overset{\circ}{a}_{\tilde{I}\tilde{J}}$ both depend on the scalar fields $\varphi^{\tilde{x}}$. The completely symmetric tensor $C_{\tilde{I}\tilde{J}\tilde{K}}$, by contrast, is constant.

The entire $\mathcal{N}=2$ MESGT (including the fermionic terms and the supersymmetry transformation laws that we have suppressed) is uniquely determined by $C_{\tilde{I}\tilde{J}\tilde{K}}$ [14]. More explicitly, $C_{\tilde{I}\tilde{J}\tilde{K}}$ defines a cubic polynomial, $\mathcal{V}(h)$, in $(\tilde{n}+1)$ real variables $h^{\tilde{I}}$ $(\tilde{I}=1,\ldots,\tilde{n}+1)$,

$$\mathcal{V}(h) := C_{\tilde{I}\tilde{J}\tilde{K}}h^{\tilde{I}}h^{\tilde{J}}h^{\tilde{K}} \ . \tag{2.2}$$

This polynomial defines a metric, $a_{\tilde{I}\tilde{J}}$, in the (auxiliary) space $\mathbb{R}^{(\tilde{n}+1)}$ spanned by the $h^{\tilde{I}}$:

$$a_{\tilde{I}\tilde{J}}(h) := -\frac{1}{3}\frac{\partial}{\partial h^{\tilde{I}}}\frac{\partial}{\partial h^{\tilde{J}}}\ln\mathcal{V}(h) \ . \tag{2.3}$$

The $\tilde{n}$-dimensional target space, $\mathcal{M}^{(5)}$, of the scalar fields $\varphi^{\tilde{x}}$ can then be represented as the hypersurface [14]

$$\mathcal{V}(h) = C_{\tilde{I}\tilde{J}\tilde{K}}h^{\tilde{I}}h^{\tilde{J}}h^{\tilde{K}} = 1 \ , \tag{2.4}$$

with $g_{\tilde{x}\tilde{y}}$ being the pull-back of (2.3) to $\mathcal{M}^{(5)}$:

$$g_{\tilde{x}\tilde{y}}(\varphi) = \frac{3}{2}(\partial_{\tilde{x}}h^{\tilde{I}})(\partial_{\tilde{y}}h^{\tilde{J}})a_{\tilde{I}\tilde{J}}|_{\mathcal{V}=1} \ . \tag{2.5}$$

Finally, the quantity $\overset{\circ}{a}_{\tilde{I}\tilde{J}}(\varphi)$ appearing in (2.1), is given by the componentwise restriction of $a_{\tilde{I}\tilde{J}}$ to $\mathcal{M}^{(5)}$:

$$\overset{\circ}{a}_{\tilde{I}\tilde{J}}(\varphi) = a_{\tilde{I}\tilde{J}}|_{\mathcal{V}=1} \ .$$

3. Charged tensor fields in five dimensions

In the previous section, we considered 5D, $\mathcal{N}=2$ ungauged supergravity theories, in which all potential tensor fields can be dualized to vector fields, and the whole theory can be expressed in terms of vector fields only. In the presence of gauge interactions, however, this equivalence between vector and tensor fields generally breaks down, and one carefully has to distinguish between them [13].

In the Maxwell-Einstein supergravity theories of the previous section, there are, in principle, the options for two types of possible gauge groups. One type corresponds to the gauging of a subgroup of the R-symmetry group, $\mathrm{SU}(2)_R$, which acts on the index i of the fermions. This type of gauging is irrelevant for the present analysis and will no longer be considered in this paper, except for a brief mentioning in section 8. The other type of gauging correspond to gaugings of symmetries of the tensor $C_{\tilde{I}\tilde{J}\tilde{K}}$. As $C_{\tilde{I}\tilde{J}\tilde{K}}$ determines the entire supergravity theory, such symmetries, if they exist, are automatically symmetries of the whole Lagrangian, and in particular, they are isometries of the scalar manifold $\mathcal{M}^{(5)}$. We denote by G the group of linear transformations of the $h^{\tilde{I}}$ and $\hat{A}_{\hat{\mu}}^{\tilde{I}}$ that leave the tensor $C_{\tilde{I}\tilde{J}\tilde{K}}$ invariant. They are generated by infinitesimal transformations of the form

$$h^{\tilde{I}} \to M_{(r)\,\tilde{J}}^{\tilde{I}} h^{\tilde{J}}, \quad \hat{A}_{\hat{\mu}}^{\tilde{I}} \to M_{(r)\,\tilde{J}}^{\tilde{I}} \hat{A}_{\hat{\mu}}^{\tilde{J}} \tag{3.1}$$

with

$$M_{(r)(\tilde{I}}^{\tilde{I}'} C_{\tilde{J}\tilde{K})\tilde{I}'} = 0 \ .$$

Here, $r=1,\ldots,\dim(G)$ counts the generators of G.

In order to turn a subgroup $K \subset G$ into a *local* (i.e., Yang-Mills-type) gauge symmetry, the $(\tilde{n}+1)$-dimensional representation of G defined by the action (3.1) has to contain the adjoint representation of K as a subrepresentation. If this is the case, there are two possibilities:

(i) The decomposition of the $(\tilde{n}+1)$-dimensional representation of G with respect to K is completely reducible.

(ii) The decomposition of the $(\tilde{n}+1)$-dimensional representation of G with respect to K is reducible, but *not* completely reducible.

Case (i), which is always the case for all connected semisimple and for all compact gauge groups, was analyzed in [13]. The second possibility (ii) has been later studied in [29]. We will first consider the first case (i), and later comment on the second case in section 6.3.

If the $(\tilde{n}+1)$-dimensional representation of G is completely reducible, the vector fields $\hat{A}_{\hat{\mu}}^{\tilde{I}}$ decompose into a direct sum of vector fields $\hat{A}_{\hat{\mu}}^{I}$ $(I=1,\ldots,n_V=\dim K)$ in the adjoint of $K \subset G$ plus possible additional non-singlets $\hat{A}_{\hat{\mu}}^{M}$ $(M=1,\ldots,n_T=(\tilde{n}+1-n_V))$ of K.[4] In order for the gauging of K to be possible, the non-singlet vectors $\hat{A}_{\hat{\mu}}^{M}$ have to be

[4] If there are also singlets of K in the $(\tilde{\mathbf{n}}+\mathbf{1})$ of G, we include them in the set of vector fields in the adjoint of (an appropriately enlarged) K, where they simply correspond to Abelian factors under which nothing is charged.

converted to antisymmetric tensor fields $\hat{B}^M_{\hat{\mu}\hat{\nu}}$ prior to the gauging [13]. We denote by f^K_{IJ} the structure constants of the gauge group $K \subset G$ and use Λ^N_{IM} for the K-transformation matrices of the tensor fields $\hat{B}^M_{\hat{\mu}\hat{\nu}}$. The transformation matrices Λ^N_{IM} of the tensor fields have to be symplectic with respect to an antisymmetric metric Ω_{MN}:

$$\Lambda^N_{IM}\Omega_{NP} + \Lambda^N_{IP}\Omega_{MN} = 0 \tag{3.2}$$

and are related to the coefficients C_{IMN} of the $C_{\tilde{I}\tilde{J}\tilde{K}}$ tensor via

$$\Lambda^N_{IM} = \frac{2}{\sqrt{6}}\Omega^{NP}C_{IMP} \Longleftrightarrow C_{IMN} = \frac{\sqrt{6}}{2}\Omega_{MP}\Lambda^P_{IN}, \tag{3.3}$$

where $\Omega_{MN}\Omega^{NP} = \delta^P_M$.

The transformation matrices $M^{\tilde{J}}_{(I)\tilde{K}}$ of eq. (3.1) that correspond to the subgroup $K \subset G$ consequently decompose as follows

$$M^{\tilde{J}}_{(I)\tilde{K}} = \begin{pmatrix} f^J_{IK} & 0 \\ 0 & \Lambda^N_{IM} \end{pmatrix}. \tag{3.4}$$

Denoting the K gauge coupling constant by g, the Yang-Mills field strengths $\hat{\mathcal{F}}^I_{\hat{\mu}\hat{\nu}}$ read

$$\hat{\mathcal{F}}^I_{\hat{\mu}\hat{\nu}} \equiv 2\partial_{[\hat{\mu}}\hat{A}^I_{\hat{\nu}]} + gf^I_{JK}\hat{A}^J_{\hat{\mu}}\hat{A}^K_{\hat{\nu}}, \tag{3.5}$$

and the covariant derivatives of the tensor fields are defined by

$$\hat{\mathcal{D}}_{[\hat{\mu}}\hat{B}^M_{\hat{\nu}\hat{\rho}]} \equiv \partial_{[\hat{\mu}}\hat{B}^M_{\hat{\nu}\hat{\rho}]} + g\hat{A}^I_{[\hat{\mu}}\Lambda^M_{IN}\hat{B}^N_{\hat{\nu}\hat{\rho}]}. \tag{3.6}$$

It is sometimes useful to combine the field strengths $\hat{\mathcal{F}}^I_{\hat{\mu}\hat{\nu}}$ and the tensor fields $\hat{B}^M_{\hat{\mu}\hat{\nu}}$ into an $(\tilde{n}+1)$-plet of two-forms,

$$\hat{\mathcal{H}}^{\tilde{I}}_{\hat{\mu}\hat{\nu}} := (\hat{\mathcal{F}}^I_{\hat{\mu}\hat{\nu}}, \hat{B}^M_{\hat{\mu}\hat{\nu}}) \tag{3.7}$$

Using the K-covariant derivative of the scalars given by

$$\hat{\mathcal{D}}_{\hat{\mu}}\varphi^{\tilde{x}} \equiv \partial_{\hat{\mu}}\varphi^{\tilde{x}} + g\hat{A}^I_{\hat{\mu}}K^{\tilde{x}}_I, \tag{3.8}$$

where $K^{\tilde{x}}_I$ denotes the Killing vectors on $\mathcal{M}^{(5)}$ that correspond to $K \subset G$, the bosonic part of the Lagrangian then reads [13]

$$\begin{aligned}
\mathcal{L}^{(5)} = &-\frac{1}{2}\hat{e}\hat{R} - \frac{1}{4}\hat{e}\overset{\circ}{a}_{\tilde{I}\tilde{J}}\hat{\mathcal{H}}^{\tilde{I}}_{\hat{\mu}\hat{\nu}}\hat{\mathcal{H}}^{\tilde{J}\hat{\mu}\hat{\nu}} - \frac{\hat{e}}{2}g_{\tilde{x}\tilde{y}}(\hat{\mathcal{D}}_{\hat{\mu}}\varphi^{\tilde{x}})(\hat{\mathcal{D}}^{\hat{\mu}}\varphi^{\tilde{y}}) \\
&+\frac{1}{6\sqrt{6}}C_{IJK}\hat{\epsilon}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\lambda}}\Big\{\hat{F}^I_{\hat{\mu}\hat{\nu}}\hat{F}^J_{\hat{\rho}\hat{\sigma}}\hat{A}^K_{\hat{\lambda}} + \frac{3}{2}g\hat{F}^I_{\hat{\mu}\hat{\nu}}\hat{A}^J_{\hat{\rho}}(f^K_{LF}\hat{A}^L_{\hat{\sigma}}\hat{A}^F_{\hat{\lambda}}) \\
&\qquad +\frac{3}{5}g^2(f^J_{GH}\hat{A}^G_{\hat{\nu}}\hat{A}^H_{\hat{\rho}})(f^K_{LF}\hat{A}^L_{\hat{\sigma}}\hat{A}^F_{\hat{\lambda}})\hat{A}^I_{\hat{\mu}}\Big\} \\
&+\frac{1}{4g}\hat{\epsilon}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\lambda}}\Omega_{MN}\hat{B}^M_{\hat{\mu}\hat{\nu}}\hat{\mathcal{D}}_{\hat{\rho}}\hat{B}^N_{\hat{\sigma}\hat{\lambda}} - \hat{e}g^2P^{(T)}.
\end{aligned} \tag{3.9}$$

Here, the scalar potential $P(T)$ is given by

$$P^{(T)} = \frac{9}{8}\overset{\circ}{a}_{MN}(\Lambda^M_{JP}h^Jh^P)(\Lambda^N_{IQ}h^Ih^Q). \tag{3.10}$$

4. The dimensional reduction to four dimensions

In this section, we dimensionally reduce the theories described in the previous sections to four dimensions. For the sake of clarity, and to set up our notation, let us first recapitulate the dimensional reduction of the *ungauged* MESGTs without tensor fields of section 2.

4.1 The ungauged MESGTs and (very) special Kähler geometry

The dimensional reduction of the bosonic sector of 5D, $\mathcal{N}=2$ MESGTs to four dimensions was first carried out in [14] and further studied in [36].

4.1.1 The reduced action

We split the fünfbein as follows

$$\hat{e}_{\hat{\mu}}^{\hat{m}} = \begin{pmatrix} e^{-\frac{\sigma}{2}} e_{\mu}^{m} & 2W_{\mu} e^{\sigma} \\ e_{5}^{m} = 0 & e^{\sigma} \end{pmatrix}, \tag{4.1}$$

which implies $\hat{e} = e^{-\sigma} e$, where $e = \det(e_{\mu}^{m})$. The Abelian field strength of W_{μ} will be denoted by $W_{\mu\nu}$:

$$W_{\mu\nu} \equiv 2\partial_{[\mu} W_{\nu]}. \tag{4.2}$$

The vector fields $\hat{A}_{\hat{\mu}}^{\tilde{I}}$ are decomposed into a 4D vector field, $A_{\mu}^{\tilde{I}}$, and a 4D scalar, $A^{\tilde{I}}$, via

$$\hat{A}_{\hat{\mu}}^{\tilde{I}} = \begin{pmatrix} \hat{A}_{\mu}^{\tilde{I}} \\ \hat{A}_{5}^{\tilde{I}} \end{pmatrix} = \begin{pmatrix} A_{\mu}^{\tilde{I}} + 2W_{\mu} A^{\tilde{I}} \\ A^{\tilde{I}} \end{pmatrix}. \tag{4.3}$$

In the following, all 4D Abelian field strengths $F_{\mu\nu}^{\tilde{I}}$ refer to $A_{\mu}^{\tilde{I}}$, which is the invariant combination with respect to the U(1) from the compactified circle:

$$F_{\mu\nu}^{\tilde{I}} \equiv 2\partial_{[\mu} A_{\nu]}^{\tilde{I}}. \tag{4.4}$$

The dimensionally reduced action of the ungauged theory (i.e., of eq. (2.1)) is then

$$\begin{aligned} e^{-1}\mathcal{L}^{(4)} = & -\frac{1}{2}R - \frac{1}{2}e^{3\sigma} W_{\mu\nu} W^{\mu\nu} - \frac{3}{4}\partial_{\mu}\sigma\partial^{\mu}\sigma \\ & -\frac{1}{4}e^{\sigma} \overset{\circ}{a}_{\tilde{I}\tilde{J}} (F_{\mu\nu}^{\tilde{I}} + 2W_{\mu\nu} A^{\tilde{I}})(F^{\tilde{J}\mu\nu} + 2W^{\mu\nu} A^{\tilde{J}}) \\ & -\frac{1}{2}e^{-2\sigma} \overset{\circ}{a}_{\tilde{I}\tilde{J}} (\partial_{\mu} A^{\tilde{I}})(\partial^{\mu} A^{\tilde{J}}) - \frac{3}{4}\overset{\circ}{a}_{\tilde{I}\tilde{J}} (\partial_{\mu} h^{\tilde{I}})(\partial^{\mu} h^{\tilde{J}}) \\ & +\frac{e^{-1}}{2\sqrt{6}} C_{\tilde{I}\tilde{J}\tilde{K}} \epsilon^{\mu\nu\rho\sigma} \Big\{ F_{\mu\nu}^{\tilde{I}} F_{\rho\sigma}^{\tilde{J}} A^{\tilde{K}} + 2F_{\mu\nu}^{\tilde{I}} W_{\rho\sigma} A^{\tilde{J}} A^{\tilde{K}} \\ & \qquad +\frac{4}{3} W_{\mu\nu} W_{\rho\sigma} A^{\tilde{I}} A^{\tilde{J}} A^{\tilde{K}} \Big\} \end{aligned} \tag{4.5}$$

4.1.2 (Very) special Kähler geometry

This can be recast in the form of special Kähler geometry (in fact, "very special" Kähler geometry in the terminology introduced in [37]) as follows [14]. Define complex coordinates

$$z^{\tilde{I}} := \frac{1}{\sqrt{3}} A^{\tilde{I}} + \frac{i}{\sqrt{2}} \tilde{h}^{\tilde{I}} \tag{4.6}$$

where

$$\tilde{h}^{\tilde{I}} := e^{\sigma} h^{\tilde{I}}. \tag{4.7}$$

These $(\tilde{n}+1)$ complex coordinates $z^{\tilde{I}}$ can be interpreted as the inhomogeneous coordinates corresponding to the $(\tilde{n}+2)$-dimensional complex vector

$$X^A = \begin{pmatrix} X^0 \\ X^{\tilde{I}} \end{pmatrix} = \begin{pmatrix} 1 \\ z^{\tilde{I}} \end{pmatrix}. \tag{4.8}$$

Introducing the "prepotential"

$$F(X^A) = -\frac{\sqrt{2}}{3} C_{\tilde{I}\tilde{J}\tilde{K}} \frac{X^{\tilde{I}} X^{\tilde{J}} X^{\tilde{K}}}{X^0} \tag{4.9}$$

and the symplectic section[5]

$$\begin{pmatrix} X^A \\ F_A \end{pmatrix} \equiv \begin{pmatrix} X^A \\ \partial_A F \end{pmatrix} , \tag{4.10}$$

one can define a Kähler potential

$$K(X(z), \bar{X}(\bar{z})) := -\ln[i\bar{X}^A F_A - iX^A \bar{F}_A] \tag{4.11}$$

$$= -\ln\left[i\frac{\sqrt{2}}{3} C_{\tilde{I}\tilde{J}\tilde{K}} (z^{\tilde{I}} - \bar{z}^{\tilde{I}})(z^{\tilde{J}} - \bar{z}^{\tilde{J}})(z^{\tilde{K}} - \bar{z}^{\tilde{K}}) \right] \tag{4.12}$$

and a "period matrix"

$$\mathcal{N}_{AB} := \bar{F}_{AB} + 2i \frac{\mathrm{Im}(F_{AC}) \mathrm{Im}(F_{BD}) X^C X^D}{\mathrm{Im}(F_{CD}) X^C X^D}, \tag{4.13}$$

where $F_{AB} \equiv \partial_A \partial_B F$ etc. The particular ("very special") form (4.9) of the prepotential leads to

$$g_{\tilde{I}\bar{\tilde{J}}} \equiv \partial_{\tilde{I}} \partial_{\bar{\tilde{J}}} K = \frac{3}{2} e^{-2\sigma} \overset{\circ}{a}_{\tilde{I}\tilde{J}} \tag{4.14}$$

for the Kähler metric, $g_{\tilde{I}\bar{\tilde{J}}}$, on the scalar manifold $\mathcal{M}^{(4)}$ of the four-dimensional theory, and

$$\mathcal{N}_{00} = -\frac{2\sqrt{2}}{9\sqrt{3}} C_{\tilde{I}\tilde{J}\tilde{K}} A^{\tilde{I}} A^{\tilde{J}} A^{\tilde{K}} - \frac{i}{3} \left(e^{\sigma} \overset{\circ}{a}_{\tilde{I}\tilde{J}} A^{\tilde{I}} A^{\tilde{J}} + \frac{1}{2} e^{3\sigma} \right)$$

[5] One should perhaps emphasize that, fundamentally, the Lagrangian can be expressed in terms of a symplectic section (X^A, F_A) without direct reference to a prepotential. In fact, a generic symplectic section need not be such that $F_A = \partial_A F$ for some function F. However, one can always go to a symplectic basis where the new F_A is, at least locally, the derivative of a prepotential F (see, e.g., [38]).

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$$
\begin{aligned}
\mathcal{N}_{0\tilde{I}} &= \frac{\sqrt{2}}{3} C_{\tilde{I}\tilde{J}\tilde{K}} A^{\tilde{J}} A^{\tilde{K}} + \frac{i}{\sqrt{3}} e^{\sigma} \overset{\circ}{a}_{\tilde{I}\tilde{J}} A^{\tilde{J}} \\
\mathcal{N}_{\tilde{I}\tilde{J}} &= -\frac{2\sqrt{2}}{\sqrt{3}} C_{\tilde{I}\tilde{J}\tilde{K}} A^{\tilde{K}} - i e^{\sigma} \overset{\circ}{a}_{\tilde{I}\tilde{J}}
\end{aligned}
\tag{4.15}
$$

for the period matrix $\mathcal{N}_{AB}$. Defining

$$
F^{0}_{\mu\nu} := -2\sqrt{3} W_{\mu\nu}, \tag{4.16}
$$

the dimensionally reduced Lagrangian (4.5) simplifies to

$$
\begin{aligned}
e^{-1}\mathcal{L}^{(4)} &= -\frac{1}{2} R - g_{\tilde{I}\bar{\tilde{J}}} (\partial_\mu z^{\tilde{I}})(\partial^\mu \bar{z}^{\tilde{J}}) \\
&\quad + \frac{1}{4} \mathrm{Im}(\mathcal{N}_{AB}) F^{A}_{\mu\nu} F^{\mu\nu B} - \frac{e^{-1}}{8} \mathrm{Re}(\mathcal{N}_{AB}) \epsilon^{\mu\nu\rho\sigma} F^{A}_{\mu\nu} F^{B}_{\rho\sigma}.
\end{aligned}
\tag{4.17}
$$

In terms of the selfdual and anti-selfdual field strengths,

$$
\begin{aligned}
F^{A\pm}_{\mu\nu} &\equiv \frac{1}{2}\left(F^{A}_{\mu\nu} \mp \frac{i}{2} e \epsilon_{\mu\nu\rho\sigma} F^{A\rho\sigma} \right) \\
F^{A\pm\mu\nu} &\equiv \frac{1}{2}\left(F^{A\mu\nu} \mp \frac{i}{2} e^{-1} \epsilon^{\mu\nu\rho\sigma} F^{A}_{\rho\sigma} \right) \ ,
\end{aligned}
\tag{4.18}
$$

where

$$
\epsilon_{\mu\nu\rho\sigma} \equiv e^{-2} \epsilon^{\lambda\kappa\eta\theta} g_{\mu\lambda} g_{\nu\kappa} g_{\rho\eta} g_{\sigma\theta} \ , \tag{4.19}
$$

the last two terms of (4.17) can also be written as

$$
\begin{aligned}
e^{-1}\mathcal{L}^{(4)\mathrm{vec}}_{\mathrm{kin}} &= \frac{1}{2} \mathrm{Im}(\mathcal{N}_{AB} F^{A+}_{\mu\nu} F^{\mu\nu B+}) \\
&\equiv -\frac{i}{4} (\mathcal{N}_{AB} F^{A+}_{\mu\nu} F^{\mu\nu B+} - \bar{\mathcal{N}}_{AB} F^{A-}_{\mu\nu} F^{\mu\nu B-}).
\end{aligned}
\tag{4.20}
$$

4.1.3 Symplectic reparametrization and global symmetries

The field strengths $F^{A+}_{\mu\nu}$ and their “duals”,

$$
G^{+}_{\mu\nu A} := \frac{\delta \mathcal{L}^{(4)}}{\delta F^{A+}_{\mu\nu}} = -\frac{ie}{2} \mathcal{N}_{AB} F^{\mu\nu B+}, \tag{4.21}
$$

can be combined into a symplectic vector

$$
\begin{pmatrix} F^{A+}_{\mu\nu} \\ G^{+}_{\mu\nu B} \end{pmatrix} \tag{4.22}
$$

so that the equations of motion that follow from (4.17) are invariant under the global symplectic rotations

$$
\begin{pmatrix} X^{A} \\ F_{B} \end{pmatrix} \longrightarrow \mathcal{O} \begin{pmatrix} X^{A} \\ F_{B} \end{pmatrix}, \qquad \begin{pmatrix} F^{A+}_{\mu\nu} \\ G^{+}_{\mu\nu B} \end{pmatrix} \longrightarrow \mathcal{O} \begin{pmatrix} F^{A+}_{\mu\nu} \\ G^{+}_{\mu\nu B} \end{pmatrix} \tag{4.23}
$$

with $\mathcal{O}$ being a symplectic matrix with respect to the symplectic metric

$$\omega = \begin{pmatrix} 0 & \delta_B{}^A \\ -\delta^C{}_D & 0 \end{pmatrix}. \tag{4.24}$$

namely $\mathcal{O}^T \omega \mathcal{O} = \omega$. Writing $\mathcal{O}$ as

$$\mathcal{O} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{4.25}$$

the period matrix $\mathcal{N}$ transforms as

$$\mathcal{N} \longrightarrow (C + D\mathcal{N})(A + B\mathcal{N})^{-1}. \tag{4.26}$$

Symplectic transformations with $B \neq 0$ correspond to non-perturbative electromagnetic duality transformations, whereas transformations with $C \neq 0$ involve shifts of the theta angles in the Lagrangian.

General symplectic tranformations will take a Lagrangian $\mathcal{L}(F, G)$ with the field strengths satisfying the Bianchi identities $dF^A = 0$ and $dG_A = 0$ to a Lagrangian $\tilde{\mathcal{L}}(\tilde{F}, \tilde{G})$ with the new field strengths satisfying $d\tilde{F}^A = 0$ and $d\tilde{G}_A = 0$.

The subgroup, $\mathcal{U}$, of $\mathrm{Sp}(2(\tilde{n}+2), \mathbb{R})$ that leaves the functional invariant

$$\tilde{\mathcal{L}}(\tilde{F}, \tilde{G}) = \mathcal{L}(\tilde{F}, \tilde{G}),$$

is called the duality invariance group (or "U-duality group"). This is a symmetry group of the equations of motion, and we will call theories related by transforations in $\mathcal{U}$ "on-shell equivalent". A subgroup of the duality invariance group that leaves the off-shell Lagrangian invariant up to surface terms is called an "electric subgroup", G_E, since it transforms electric field strengths into electric field strengths only. Obviously, we have the inclusions

$$G_E \subset \mathcal{U} \subset \mathrm{Sp}(2(\tilde{n}+2), \mathbb{R}).$$

Pure electric-magnetic exchanges are contained in the coset $\mathcal{U}/G_E$. Hodge-dualizations, contained in $\mathrm{Sp}(2\tilde{n}+4)/\mathcal{U}$ [39], lead to "dual theories" that generally have different manifest electric subgroups G_E.

A four-dimensional MESGT that derives from five dimensions with the prepotential (4.9) automatically has the following (global) duality symmetries:[6]

1. The whole global symmetry group of the 5D Lagrangian, i.e., the invariance group G of the cubic polynomial $\mathcal{V}(h) = C_{\tilde{I}\tilde{J}\tilde{K}} h^{\tilde{I}} h^{\tilde{J}} h^{\tilde{K}}$, survives as a global symmetry group of the 4D theory.

[6] There might be additional hidden symmetries for certain scalar manifolds, such as symmetric spaces [14], or some homogeneous spaces [40]. However, in general, there are no additional hidden symmetries. The number of hidden symmetry generators is maximum for symmetric target spaces in four dimensions and is equal to the number of translation (shift) generators.

2. The shifts $z^{\tilde{I}} \to z^{\tilde{I}} + b^{\tilde{I}}$ with constant real parameters $b^{\tilde{I}}$ (i.e., the shifts of the Kaluza-Klein scalars $A^{\tilde{I}}$) become symmetries of the 4D theories if they are accompanied by simultaneous transformations

$$F^{\tilde{I}}_{\mu\nu} \to F^{\tilde{I}}_{\mu\nu} - 2W_{\mu\nu}b^{\tilde{I}} \tag{4.27}$$

of the field strengths.

3. There is an additional global scaling symmetry

$$X^0 \to e^{\beta}X^0, \qquad X^{\tilde{I}} \to e^{-\frac{\beta}{3}}X^{\tilde{I}} \tag{4.28}$$

which leaves the prepotential (4.9) invariant.

Together these symmetries form the global invariance group

$$(G \times \mathrm{SO}(1,1)) \ltimes T^{(\tilde{n}+1)}, \tag{4.29}$$

where $\mathrm{SO}(1,1)$ describes the scaling symmetry, $T^{(\tilde{n}+1)}$ refers to the real translations of scalars by $b^{\tilde{I}}$, and $\ltimes$ denotes a semi-direct product. The symplectic matrix $\mathcal{O}$ that implements these symmetries on the symplectic sections (4.10), (4.22) is block diagonal for $G \times \mathrm{SO}(1,1)$, but involves shifts of the theta angles for the translations $T^{(\tilde{n}+1)}$. More precisely, an infinitesimal transformation of $(G \times \mathrm{SO}(1,1)) \ltimes T^{(\tilde{n}+1)}$ is represented by

$$\mathcal{O} = \mathbf{1} + \begin{pmatrix} B & 0 \\ C & -B^T \end{pmatrix} \tag{4.30}$$

with

$$B^A{}_B = \begin{pmatrix} \beta & 0 \\ b^{\tilde{I}} & [M_{(r)\tilde{J}}^{\tilde{I}} + \frac{1}{3}\beta\delta^{\tilde{I}}_{\tilde{J}}] \end{pmatrix}, \qquad C_{AB} = \begin{pmatrix} 0 & 0 \\ 0 & -2\sqrt{2}C_{\tilde{I}\tilde{J}\tilde{K}}b^{\tilde{K}} \end{pmatrix}, \tag{4.31}$$

where $b^{\tilde{I}}$ is now an infinitesimal shift parameter and only terms linear in the transformation parameters are kept. Note that for different symplectic sections, the above transformation matrices also change their form in general. In order to gauge symmetries in the standard way [10, 11], one works in a symplectic basis, where the symmetries one wants to gauge are represented by block-diagonal symplectic matrices. However, there are cases in which also off-diagonal transformations can be gauged by certain "non-standard" gaugings [11, 41], but often these gaugings turn out to be dual to a standard gauging. We will come back to this point later.

4.2 The dimensional reduction of $\mathcal{N} = 2$ YMESGTs with tensor fields

In this subsection, we consider the dimensional reduction of a 5D YMESGT with tensor fields to 4D. Our starting point is thus the 5D Lagrangian (3.9). Just as for the ungauged case, we decompose the fünfbein as in eq. (4.1) and the vector fields $\hat{A}^I_{\hat{\mu}}$ as in (4.3) (remembering that we now no longer have 5D vector fields with an index M, as these are converted to 5D tensor fields). The 5D tensor fields $\hat{B}^M_{\hat{\mu}\hat{\nu}}$ are decomposed into Kaluza-Klein invariant 4D tensor fields, $B^M_{\mu\nu}$, and 4D vector fields, B^M_μ:

$$\hat{B}^M_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \hat{B}^M_{\mu\nu} \\ \hat{B}^M_{\mu 5} \end{pmatrix} = \begin{pmatrix} B^M_{\mu\nu} - 4W_{[\mu}B^M_{\nu]} \\ B^M_\mu \end{pmatrix}. \tag{4.32}$$

As is outlined in appendix A, this results in the 4D Lagrangian

$$\begin{aligned} e^{-1}\mathcal{L}^{(4)} = & -\frac{1}{2}R - \frac{3}{4}\overset{\circ}{a}_{\tilde{I}\tilde{J}}(\mathcal{D}_\mu \tilde{h}^{\tilde{I}})(\mathcal{D}^\mu \tilde{h}^{\tilde{J}}) - \frac{1}{2}e^{-2\sigma}\overset{\circ}{a}_{IJ}(\mathcal{D}_\mu A^I)(\mathcal{D}^\mu A^J) \\ & -e^{-2\sigma}\overset{\circ}{a}_{IM}(\mathcal{D}_\mu A^I)B^{\mu M} - \frac{1}{2}e^{-2\sigma}\overset{\circ}{a}_{MN}B^M_\mu B^{\mu N} \\ & +\frac{e^{-1}}{g}\epsilon^{\mu\nu\rho\sigma}\Omega_{MN}B^M_{\mu\nu}(\partial_\rho B^N_\sigma + gA^I_\rho \Lambda^N_{IP}B^P_\sigma) \\ & +\frac{e^{-1}}{g}\epsilon^{\mu\nu\rho\sigma}\Omega_{MN}W_{\mu\nu}B^M_\rho B^N_\sigma + \frac{e^{-1}}{2\sqrt{6}}C_{MNI}\epsilon^{\mu\nu\rho\sigma}B^M_{\mu\nu}B^N_{\rho\sigma}A^I \\ & -\frac{1}{4}e^\sigma \overset{\circ}{a}_{MN}B^M_{\mu\nu}B^{N\mu\nu} - \frac{1}{2}e^\sigma\overset{\circ}{a}_{IM}(\mathcal{F}^I_{\mu\nu} + 2W_{\mu\nu}A^I)B^{M\mu\nu} \\ & -\frac{1}{4}e^\sigma\overset{\circ}{a}_{IJ}(\mathcal{F}^I_{\mu\nu} + 2W_{\mu\nu}A^I)(\mathcal{F}^{J\mu\nu} + 2W^{\mu\nu}A^J) - \frac{1}{2}e^{3\sigma}W_{\mu\nu}W^{\mu\nu} \\ & +\frac{e^{-1}}{2\sqrt{6}}C_{IJK}\epsilon^{\mu\nu\rho\sigma}\left\{\mathcal{F}^I_{\mu\nu}\mathcal{F}^J_{\rho\sigma}A^K + 2\mathcal{F}^I_{\mu\nu}W_{\rho\sigma}A^JA^K + \frac{4}{3}W_{\mu\nu}W_{\rho\sigma}A^IA^JA^K\right\} \\ & -g^2 P, \end{aligned} \tag{4.33}$$

where

$$\mathcal{D}_\mu A^I \equiv \partial_\mu A^I + gA^J_\mu f^I_{JK}A^K \tag{4.34}$$

$$\mathcal{F}^I_{\mu\nu} \equiv 2\partial_{[\mu}A^I_{\nu]} + gf^I_{JK}A^J_\mu A^K_\nu \tag{4.35}$$

$$\mathcal{D}_\mu \tilde{h}^{\tilde{I}} \equiv \partial_\mu \tilde{h}^{\tilde{I}} + gA^I_\mu M^{\tilde{I}}_{I\tilde{K}}\tilde{h}^{\tilde{K}}, \tag{4.36}$$

and the total scalar potential, P, is given by

$$P = e^{-\sigma}P^{(T)} + \frac{3}{4}e^{-3\sigma}\overset{\circ}{a}_{\tilde{I}\tilde{J}}(A^I M^{\tilde{I}}_{I\tilde{K}}h^{\tilde{K}})(A^J M^{\tilde{J}}_{J\tilde{L}}h^{\tilde{L}}), \tag{4.37}$$

Note that, in the first line of (4.33), we have absorbed the kinetic term of sigma by defining $\tilde{h}^{\tilde{I}}$ as in (4.7).

This Lagrangian has several interesting features:

- Whereas the scalars $h^{\tilde{I}}$ are complete, the scalars A^M one had in the ungauged theory, have disappeared from the Lagrangian. This was to be expected, as the scalars A^M in the ungauged theory have their origin in the 5D vector fields $\hat{A}^M_{\hat{\mu}}$, which, however, are dualized to the 5D two-form fields $\hat{B}^M_{\hat{\mu}\hat{\nu}}$ in the gauged version, and the $\hat{B}^M_{\hat{\mu}\hat{\nu}}$ do not give rise to 4D scalar fields.
- The terms in the second line of (4.33) suggest that the scalar A^M has been eaten by the vector fields B^M_μ as the result of a Peccei-Quinn-type gauging of the translations

$A^M \to A^M + b^M$ (cf. section 4.1.3). In the standard symplectic basis, however, the shifts of the $A^{\tilde{I}}$ are not blockdiagonal symplectic transformations (see eq. (4.31)). The conventional gauging of isometries of the scalar manifold described in [10, 11] requires a blockdiagonal embedding of the isometries in the corresponding symplectic duality matrices. The theory at hand can therefore be interpreted in either of two ways: either as a *non*-standard gauging in the "conventional" symplectic basis, or as a standard gauging in a rotated symplectic section. We will map the above theory to such a standard gauging below.

- "Regurgitating" scalar fields A^M from the B_μ^M, or, more precisely, making the replacement

$$B_\mu^M \to gB_\mu^M + \mathcal{D}_\mu A^M \tag{4.38}$$

together with the shift

$$B_{\mu\nu}^M \to gB_{\mu\nu}^M + F_{\mu\nu}^M + 2W_{\mu\nu}A^M, \tag{4.39}$$

and switching off g, leads back to the ungauged theory (4.5).

- After having eaten the scalar fields A^M, the vector fields B_μ^M acquire a mass term (the last term in the second line of (4.33)). However, there is no explicit *kinetic* term for the B_μ^M. Instead, there are the two-form fields $B_{\mu\nu}^M$, which also have a mass term (in line 5 of (4.33)), but also no second order kinetic term. The two-forms have a one derivative interaction with the vectors B_μ^M in the third line of (4.33). Such a term normally allows the elimination of the tensor fields in favor of the vector fields or vice versa, so that one either obtains massive vector fields with a standard second order kinetic term or massive tensor fields with a standard second order kinetic term. This is possible because massive vectors are dual to massive tensors in 4D. As we will show below, it is indeed possible to eliminate the tensors $B_{\mu\nu}^M$ in favor of the vectors B_μ^M. However, the converse seems to be difficult, if not impossible to achieve locally, due to the term proportional to $\epsilon^{\mu\nu\rho\sigma}W_{\mu\nu}B_\rho^M B_\sigma^N$ in the fourth line in (4.33).

- The tensors $B_{\mu\nu}^M$ and the vectors B_μ^M are charged under the 5D gauge group, K, which also descends to a local gauge symmetry in 4D. This is in contrast to the massive tensor fields that have been recently considered in the literature [2–4]. The tensor fields in those papers arise from dualizations of scalars of the quaternionic manifold instead of the special Kähler manifold and also don't carry any charge with respect to a non-trivial local gauge group. The Lagrangian (4.33) does, however, have some resemblance with the Freedman-Townsend model [30] (see also [12]).

- The terms in the sixth and seventh line of (4.33) can be written as

$$\frac{1}{2}\mathrm{Im}\Big[\mathcal{N}_{00}F_{\mu\nu}^{0+}F^{\mu\nu 0+} + 2\mathcal{N}_{0I}F_{\mu\nu}^{0+}\mathcal{F}^{\mu\nu I+} + \mathcal{N}_{IJ}\mathcal{F}_{\mu\nu}^{I+}\mathcal{F}^{\mu\nu I+}\Big]\Big|_{A^M=0}. \tag{4.40}$$

5. Eliminating the tensor fields

The action (4.33) contains the terms

$$-\frac{1}{4}e^{\sigma}\overset{\circ}{a}_{MN}B^M_{\mu\nu}B^{N\mu\nu} - \frac{1}{2}e^{-2\sigma}\overset{\circ}{a}_{MN}B^M_{\mu}B^{\mu N} + \frac{e^{-1}}{g}\epsilon^{\mu\nu\rho\sigma}\Omega_{MN}B^M_{\mu\nu}\partial_\rho B^N_\sigma \tag{5.1}$$

If these were the only terms involving $B^M_{\mu\nu}$ and B^N_μ, one could simply, as mentioned in the previous section, integrate out $B^M_{\mu\nu}$ in favor of B^N_μ, which, schematically, would result in a relation of the form

$$B^{\mu\nu M} = \mathcal{T}^M{}_N e^{-1}\epsilon^{\mu\nu\rho\sigma}\partial_\rho B^N_\sigma \tag{5.2}$$

with some matrix $\mathcal{T}^M{}_N$ and leads to a standard second order action for massive vector fields B^N_μ,

$$-\mathcal{K}_{MN}(\partial_{[\mu}B^M_{\nu]})(\partial^{[\mu}B^{\nu]N}) - \mathcal{M}_{MN}B^M_\mu B^{\mu N}, \tag{5.3}$$

with a kinetic and a mass matrix $\mathcal{K}_{MN}$ and $\mathcal{M}_{MN}$, respectively.

Alternatively, one could also choose to integrate out the vector fields in favor of the tensor fields, which then would lead to a relation of the form

$$B^{\mu M} = \tilde{\mathcal{T}}^M{}_N e^{-1}\epsilon^{\mu\nu\rho\sigma}\partial_\nu B^N_{\rho\sigma} \tag{5.4}$$

and a standard second order action for massive tensor fields,

$$-\tilde{\mathcal{K}}_{MN}(\partial_{[\mu}B^M_{\nu\rho]})(\partial^{[\mu}B^{\nu\rho]N}) - \tilde{\mathcal{M}}_{MN}B^M_{\mu\nu}B^{\mu\nu N}. \tag{5.5}$$

However, this is not quite what happens, as in the actual Lagrangian (4.33), there are also other quadratic terms of the form

$$\frac{e^{-1}}{g}\Omega_{MN}\epsilon^{\mu\nu\rho\sigma}W_{\mu\nu}B^M_\rho B^N_\sigma \tag{5.6}$$

and

$$\frac{e^{-1}}{2\sqrt{6}}C_{MNI}\epsilon^{\mu\nu\rho\sigma}B^M_{\mu\nu}B^N_{\rho\sigma}A^I. \tag{5.7}$$

The first of these two terms would contribute a term proportional to

$$e^{-1}\epsilon^{\mu\nu\rho\sigma}W_{\nu\rho}B^M_\sigma \tag{5.8}$$

to the left hand side of eq. (5.4). This additional term seems to make it impossible to (locally) eliminate the vector fields B^N_μ in favor of the tensor fields $B^M_{\mu\nu}$, as the field strength $W_{\mu\nu}$ would somehow have to be "inverted" to solve the equation for B^N_μ. The second term, (5.7), on the other hand, would yield a contribution involving

$$e^{-1}C_{MNI}A^I\epsilon^{\mu\nu\rho\sigma}B^N_{\rho\sigma} \tag{5.9}$$

to the left hand side of (5.2). This involves only scalar fields in front of $B^M_{\mu\nu}$, which, in principle, can be inverted so as to solve the modified eq. (5.2) for $B^M_{\mu\nu}$. Due to the epsilon

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tensor, however, one has to switch to the selfdual and anti-selfdual components of all two-forms. In addition, there are also further terms linear in the $B^M_{\mu\nu}$ in eq. (4.33), which we have neglected in the above schematic discussion. Let us therefore become more specific now and carry out the elimination of the tensor fields in detail. To this end, we write the $B^M_{\mu\nu}$-dependent terms in (4.33) as follows

$$e^{-1}\mathcal{L}^{(4)}_{B^M_{\mu\nu}} = \frac{1}{2}\mathrm{Im}\Big[\mathcal{N}_{MN}B^{M+}_{\mu\nu}B^{\mu\nu N+}\Big]\Big|_{A^M=0} + 2\mathrm{Re}\Big[J^{+\mu\nu}_M B^{M+}_{\mu\nu}\Big], \tag{5.10}$$

where we have introduced

$$J^{\mu\nu}_M := -\frac{1}{2}e^{\sigma}\overset{\circ}{a}_{IM}(\mathcal{F}^{\mu\nu I} + 2W^{\mu\nu}A^I) + \frac{e^{-1}}{g}\epsilon^{\mu\nu\rho\sigma}\Omega_{MN}\mathcal{D}_\rho B^N_\sigma \tag{5.11}$$

and used

$$\mathcal{N}_{MN}\big|_{A^M=0} = -\frac{4}{\sqrt{6}}C_{MNI}A^I - ie^{\sigma}\overset{\circ}{a}_{MN}. \tag{5.12}$$

Varying (5.10) with respect to $B^M_{\mu\nu}$, one obtains

$$J^{\mu\nu+}_M = \frac{i}{2}\mathcal{N}_{MN}\big|_{A^M=0}B^{\mu\nu N+}, \tag{5.13}$$

which can be used to express $B^{M+}_{\mu\nu}$ in terms of $J^+_{\mu\nu M}$ in (5.10) with the result

$$e^{-1}\mathcal{L}^{(4)}_{B^M_{\mu\nu}} = 2\mathrm{Im}\Big[\mathcal{N}^{MN}J^+_{\mu\nu M}J^{\mu\nu+}_N\Big]\Big|_{A^M=0}. \tag{5.14}$$

Here, $\mathcal{N}^{MN}$ denotes the inverse of $\mathcal{N}_{MN}$,

$$\mathcal{N}_{MN}\mathcal{N}^{NP} = \delta^P_M. \tag{5.15}$$

The Lagrangian (4.33) now takes on a more concise form:

$$\begin{aligned} e^{-1}\mathcal{L}^{(4)} = & -\frac{1}{2}R - \frac{3}{4}\overset{\circ}{a}_{\tilde{I}\tilde{J}}(\mathcal{D}_\mu\tilde{h}^{\tilde{I}})(\mathcal{D}^\mu\tilde{h}^{\tilde{J}}) - \frac{1}{2}e^{-2\sigma}\overset{\circ}{a}_{IJ}(\mathcal{D}_\mu A^I)(\mathcal{D}^\mu A^J) \\ & -e^{-2\sigma}\overset{\circ}{a}_{IM}(\mathcal{D}_\mu A^I)B^{\mu M} - \frac{1}{2}e^{-2\sigma}\overset{\circ}{a}_{MN}B^M_\mu B^{\mu N} \\ & +\frac{1}{2}\mathrm{Im}\Big[\mathcal{N}_{00}F^{0+}_{\mu\nu}F^{\mu\nu 0+} + 2\mathcal{N}_{0I}F^{0+}_{\mu\nu}\mathcal{F}^{\mu\nu I+} + \mathcal{N}_{IJ}\mathcal{F}^{I+}_{\mu\nu}\mathcal{F}^{\mu\nu J+}\Big]\Big|_{A^M=0} \\ & +2\mathrm{Im}\Big[\mathcal{N}^{MN}J^+_{\mu\nu M}J^{\mu\nu+}_N\Big]\Big|_{A^M=0} + \frac{e^{-1}}{g}\epsilon^{\mu\nu\rho\sigma}\Omega_{MN}W_{\mu\nu}B^M_\rho B^N_\sigma \\ & -g^2 P. \end{aligned} \tag{5.16}$$

6. The equivalence to a standard gauging

In this section, we show that the above action (5.16) is dual to a standard gauged supergravity theory of the type described in [10, 11]. We already identified the translations $A^M \to b^M$ and the 5D gauge group K generated by the matrices $M^{\tilde{I}}_{(I)\tilde{J}}$ of eq. (3.4) as part of the 4D gauge group. We also saw, in (4.31), however, that, in the ungauged theory, the translations $A^{\tilde{I}} \to b^{\tilde{I}}$ are not represented by block diagonal symplectic matrices if one works in the "natural" symplectic basis

$$\begin{pmatrix} X^A \\ F_B \end{pmatrix} = \begin{pmatrix} X^A \\ \partial_B F \end{pmatrix} \tag{6.1}$$

with $X^0 = 1$ and $X^{\tilde{I}} = z^{\tilde{I}}$ and

$$\begin{pmatrix} F^A_{\mu\nu} \\ G_{\mu\nu B} \end{pmatrix}, \tag{6.2}$$

with $F^0_{\mu\nu} = -2\sqrt{3}W_{\mu\nu}$, which one directly gets from the dimensional reduction from 5D. In order to gauge the translations associated with b^M in the standard way, one therefore has to go to a different symplectic basis in which both the translations by b^M and the K transformations are represented by block diagonal symplectic matrices. To see how this works, we split the $z^{\tilde{I}}$ into (z^I, z^M) and take into account that $C_{MNP} = C_{IJM} = 0$. The symplectic vector (6.1) then becomes

$$\begin{pmatrix} X^0 \\ X^I \\ X^M \\ F_0 \\ F_I \\ F_M \end{pmatrix} = \begin{pmatrix} 1 \\ z^I \\ z^M \\ \sqrt{2}/3[C_{IJK}z^I z^J z^K + 3C_{IMN}z^I z^M z^N] \\ -\sqrt{2}[C_{IJK}z^J z^K + C_{IMN}z^M z^N] \\ -2\sqrt{2}C_{MNI}z^N z^I \end{pmatrix} \tag{6.3}$$

Under an infinitesimal translation $z^M \to z^M + b^M$, this transforms as

$$\begin{pmatrix} X^0 \\ X^I \\ X^M \\ F_0 \\ F_I \\ F_M \end{pmatrix} \to \begin{pmatrix} X^0 \\ X^I \\ X^M \\ F_0 \\ F_I \\ F_M \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ b^M X^0 \\ -b^M F_M \\ -2\sqrt{2}b^M C_{MNI}X^N \\ -2\sqrt{2}b^N C_{MIN}X^I \end{pmatrix}, \tag{6.4}$$

where we have, somewhat redundantly, inserted $X^0 = 1$ in the third line and kept only terms linear in the infinitesimal parameters b^M.

From this expression, it becomes clear that (X^0, F_I, X^M) transform among themselves, as do (F_0, X^I, F_M). Thus, a symplectic duality rotation that exchanges X^0 with F_0 and X^M with F_M, could make the translations $z^M \to z^M + b^M$ blockdiagonal. At the same time, we want this symplectic duality rotation to preserve the block diagonality of the K transformations (3.4). In our original basis (6.3), a combined infinitesimal translation $z^M \to z^M + b^M$ and infinitesimal K transformation with parameter α^I is generated by the symplectic matrix

$$\mathcal{O} = \mathbf{1} + \begin{pmatrix} B & 0 \\ C & -B^T \end{pmatrix}, \tag{6.5}$$

with

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha^I f_{IJ}^K & 0 \\ b^M & 0 & \alpha^I \Lambda_{IN}^M \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & B_{IM} \\ 0 & B_{MI} & 0 \end{pmatrix}, \tag{6.6}$$

where

$$B_{IM} := -2\sqrt{2} C_{IMN} b^N. \tag{6.7}$$

In order to get this block diagonal, we switch to a new symplectic basis

$$\begin{pmatrix} X^A \\ F_B \end{pmatrix} \to \begin{pmatrix} \check{X}^A \\ \check{F}_B \end{pmatrix} \equiv \mathcal{S} \begin{pmatrix} X^A \\ F_B \end{pmatrix}, \qquad \begin{pmatrix} F_{\mu\nu}^A \\ G_{\mu\nu B} \end{pmatrix} \to \begin{pmatrix} \check{F}_{\mu\nu}^A \\ \check{G}_{\mu\nu B} \end{pmatrix} \equiv \mathcal{S} \begin{pmatrix} F_{\mu\nu}^A \\ G_{\mu\nu B} \end{pmatrix} \tag{6.8}$$

where

$$\mathcal{S} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \delta^J{}_I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & D^{MN} \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta_I{}^J & 0 \\ 0 & 0 & D_{MN} & 0 & 0 & 0 \end{pmatrix}, \tag{6.9}$$

and

$$D_{MN} := -2\sqrt{3}\Omega_{MN}, \qquad D_{MN} D^{NP} = \delta_M^P. \tag{6.10}$$

It is easy to verify that the rotation matrix $\mathcal{S}$ is itself symplectic and that

$$\check{\mathcal{O}} \equiv \mathcal{S}\mathcal{O}\mathcal{S}^{-1} = \mathbf{1} + \begin{pmatrix} \check{B} & 0 \\ 0 & -\check{B}^T \end{pmatrix} \tag{6.11}$$

with

$$\check{B} = \begin{pmatrix} 0 & 0 & 2\sqrt{3} b^M \Omega_{MN} \\ 0 & \alpha^I f_{IJ}^K & 0 \\ 0 & +\Lambda_{IM}^N b^M & \alpha^I \Lambda_{IM}^N \end{pmatrix}. \tag{6.12}$$

Here, we have used (3.2), (3.3) and (6.10). Hence, in the new basis $(\check{X}^A, \check{F}_B)$, $(\check{F}_{\mu\nu}^A, \check{G}_{\mu\nu B})$, the group $K \ltimes \mathbb{R}^{n_T}$ is represented by block diagonal symplectic matrices $\check{\mathcal{O}}$. But this is not all; setting

$$\check{B}^C{}_B = \alpha^A f_{AB}^C, \tag{6.13}$$

one reads off

$$f_{IJ}^K, \qquad f_{IM}^N = \Lambda_{IM}^N = -f_{MI}^N, \qquad f_{MN}^0 = -2\sqrt{3}\Omega_{MN}, \tag{6.14}$$

as the non-vanishing components as well as $\alpha^M = -b^M$. It is easy to see that the non-vanishing f_{AB}^C of eq. (6.14) define the Lie algebra of a central extension of the Lie algebra of $K \ltimes \mathbb{R}^{n_T}$, with the central charge corresponding to the index 0.[7] We shall denote the

[7]Note that there is a subtlety here concerning the central charge. As one easily verifies, two translations represented by matrices of the form (6.11) and (6.12) with $\alpha^I = 0$ and two parameters b^M and $b^{M'}$, always commute, even though $f_{MN}^0 \neq 0$. However that is a generic property of finite-dimensional representations of centrally extended Lie algebras such as the Heisenberg algebra.

corresponding group of this centrally extended Lie algebra as $K \ltimes \mathcal{H}^{n_T+1}$, where $\mathcal{H}^{n_T+1}$ denotes the Heisenberg group generated by the translations and the central charge.

As the structure constants define the adjoint representation, this centrally extended group can therefore be gauged in the standard way if one uses the new symplectic basis $(\check{X}^A, \check{F}_B)$. As we will show now, the resulting Lagrangian of this $K \ltimes \mathcal{H}^{n_T+1}$ gauged theory is dual to the Lagrangian (5.16) of the previous section, which we got from the dimensional reduction of a 5D theory with tensor fields. In order to show this, we will start from the 4D ungauged theory in the new symplectic basis $(\check{X}^A, \check{F}_B)$, $(\check{F}^A_{\mu\nu}, \check{G}_{\mu\nu B})$ and assume the subsequent gauging of the group $K \ltimes \mathcal{H}^{n_T+1}$ using the standard formulae [10, 11] evaluated in that new basis. As this gauging is fairly standard, we can skip the details and immediately write down the resulting Lagrangian. This standard Lagrangian with gauge group $K \ltimes \mathcal{H}^{n_T+1}$ will then be subjected to a few field redefinitions and dualizations until it precisely coincides with the Lagrangian (5.16) from the dimensional reduction of a 5D theory with tensor fields.

We will first carry out this program for the scalar sector and after that for the kinetic terms of the vector fields.

6.1 The scalar sector

The Kähler potential $K(z, \bar{z})$ of eq. (4.11) is a symplectic invariant. Thus, the metric $g_{\tilde{I}\bar{\tilde{J}}}$ stays the same as in the old symplectic basis. The gauging of $K \ltimes \mathcal{H}^{n_T+1}$, however, leads to two modifications in the scalar sector. First, the kinetic term of the scalars becomes covariant with respect to the gauge group:

$$-g_{\tilde{I}\bar{\tilde{J}}}(\partial_\mu z^{\tilde{I}})(\partial^\mu \bar{z}^{\tilde{J}}) \rightarrow -g_{\tilde{I}\bar{\tilde{J}}}(\mathcal{D}_\mu z^{\tilde{I}})(\mathcal{D}^\mu \bar{z}^{\tilde{J}}) \tag{6.15}$$

with

$$\mathcal{D}_\mu z^{\tilde{I}} = \partial_\mu z^{\tilde{I}} + g\check{A}^A_\mu K^{\tilde{I}}_A. \tag{6.16}$$

Here, $K^{\tilde{I}}_A(z)$ are the holomorphic Killing vectors that generate the gauge group on the scalar manifold $\mathcal{M}^{(4)}$. They can be expressed in terms of derivatives of Killing prepotentials, P_A,

$$K^{\tilde{I}}_A = ig^{\tilde{I}\bar{\tilde{J}}}\partial_{\bar{\tilde{J}}} P_A, \tag{6.17}$$

where [10, 11]

$$P_A = e^K(\check{F}_B f^B_{AC} \bar{\check{X}}^C + \bar{\check{F}}_B f^B_{AC} \check{X}^C). \tag{6.18}$$

Using this, one obtains

$$\begin{aligned} P_0 &= 0 \\ P_I &= -\sqrt{2}e^K\left(C_{\tilde{I}\tilde{J}\tilde{K}} z^{\tilde{J}} z^{\tilde{K}} M^{\tilde{I}}_{(I)\tilde{L}} \bar{z}^{\tilde{L}}\right) + \text{c.c.} \\ P_M &= -2\sqrt{2}e^K C_{IMP}\left(z^P \bar{z}^I - \bar{z}^P \bar{z}^I\right) + \text{c.c.} \end{aligned} \tag{6.19}$$

and then

$$K^{\tilde{J}}_0 = 0$$

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$$\begin{aligned} K_I^{\tilde{J}} &= M^{\tilde{J}}_{(I)\tilde{L}} z^{\tilde{L}} \\ K_M^{\tilde{I}} &= -\delta^{\tilde{I}}_M. \end{aligned} \tag{6.20}$$

Upon the identification

$$\begin{aligned} A^I_\mu &= \check{A}^I_\mu \\ B^M_\mu &= -g\sqrt{3}\check{A}^M_\mu, \end{aligned} \tag{6.21}$$

the kinetic term of the scalars then becomes

$$-g_{\tilde{I}\bar{\tilde{J}}}(\mathcal{D}_\mu z^{\tilde{I}})(\mathcal{D}^\mu \bar{z}^{\tilde{J}}) = -\frac{3}{4}\overset{\circ}{a}_{\tilde{I}\tilde{J}}(\mathcal{D}_\mu \tilde{h}^{\tilde{I}})(\mathcal{D}^\mu \tilde{h}^{\tilde{J}}) - \frac{1}{2}e^{-2\sigma}\overset{\circ}{a}_{\tilde{I}\tilde{J}}(\mathcal{D}'_\mu A^{\tilde{I}})(\mathcal{D}'^\mu A^{\tilde{J}}) \tag{6.22}$$

with

$$\begin{aligned} \mathcal{D}_\mu \tilde{h}^{\tilde{I}} &= \partial_\mu \tilde{h}^{\tilde{I}} + gA^I_\mu M^{\tilde{I}}_{(I)\tilde{K}}\tilde{h}^{\tilde{K}} \\ \mathcal{D}'_\mu A^{\tilde{I}} &= \partial_\mu A^{\tilde{I}} + gA^I_\mu M^{\tilde{I}}_{(I)\tilde{K}} A^{\tilde{K}} + B^M_\mu \delta^{\tilde{I}}_M. \end{aligned} \tag{6.23}$$

The vector fields B^M_μ can now absorb the scalars A^M, as anticipated, and, after also adding the gravitational term, we have reproduced the first two lines of (5.16).[8]

The gauging also induces a second contribution to the scalar sector, namely a scalar potential. Using the standard expressions, this scalar potential should be

$$V = e^K(\check{X}^A \bar{K}^{\tilde{I}}_A) g_{\tilde{I}\bar{\tilde{J}}}(\bar{\check{X}}^B K^{\tilde{J}}_B). \tag{6.24}$$

Using (6.20) and expressing the $\check{X}^{\tilde{I}}$ in terms of the $z^{\tilde{I}}$, one finds that $V = P$, where P is the potential (4.37) of the dimensionally reduced Lagrangian (5.16). Thus, the two scalar sectors completely agree. It remains to verify the agreement for the kinetic terms of the vector fields.

6.2 The kinetic terms of the vector fields

We shall now compare kinetic terms of the vector fields of (5.16) with those of the $K \ltimes \mathcal{H}^{n_T+1}$ gauged theory. By kinetic terms of the vector fields, we mean the terms in the third and fourth line of (5.16), which, using (4.16), (6.21) and (6.14), can be rewritten as

$$\begin{aligned} e^{-1}\mathcal{L}^{(4)\mathrm{kin}}_{\mathrm{vec}} &= \frac{1}{2}\mathrm{Im}\Big[\mathcal{N}_{00}F^{0+}_{\mu\nu}F^{\mu\nu 0+} + 2\mathcal{N}_{0I}F^{0+}_{\mu\nu}\check{\mathcal{F}}^{\mu\nu I+} + \mathcal{N}_{IJ}\check{\mathcal{F}}^{I+}_{\mu\nu}\check{\mathcal{F}}^{\mu\nu I+}\Big]\Big|_{A^M=0} \\ &\quad +2\mathrm{Im}\Big[\mathcal{N}^{MN}J^+_{\mu\nu M}J^{\mu\nu +}_N\Big]\Big|_{A^M=0} - \mathrm{Im}\Big[F^{0+}_{\mu\nu}Z^{\mu\nu +}\Big], \end{aligned} \tag{6.25}$$

[8] We should perhaps emphasize that here we are discussing the gauging of the real translational isometries (of $Re(z^M)$). The resulting massive BPS vector supermultiplets have scalars given by $Im(z^M)$. This is to be contrasted with the dimensional reduction of 5D YMESGTs with noncompact gauge groups, in which the 4D vector fields associated with noncompact symmetries belong to massive BPS supermultiplets whose scalar fields are $Re(z^M)$. This is best seen by the fact that in five dimensions the non-compact gauge fields become massive by eating the scalars which in four dimensions correspond to the imaginary part of z^M.

with

$$Z_{\mu\nu} := g f^0_{MN} \check{A}^M_\mu \check{A}^N_\nu = -\frac{2}{\sqrt{3}g} \Omega_{MN} B^M_\mu B^N_\nu \tag{6.26}$$

$$J^{\mu\nu}_M \equiv -\frac{1}{2} e^\sigma \overset{\circ}{a}_{IM} \Big(\check{\mathcal{F}}^{\mu\nu I} - \frac{1}{\sqrt{3}} F^{\mu\nu 0} A^I \Big) - \sqrt{3} e^{-1} \epsilon^{\mu\nu\rho\sigma} \Omega_{MN} \mathcal{D}_\rho \check{A}^N_\sigma . \tag{6.27}$$

Using (cf. eq. (4.15))

$$\mathcal{N}_{IM}\big|_{A^M=0} = -i e^\sigma \overset{\circ}{a}_{IM} \tag{6.28}$$

$$\mathcal{N}_{0M}\big|_{A^M=0} = \frac{i}{\sqrt{3}} e^\sigma \overset{\circ}{a}_{IM} A^I \tag{6.29}$$

as well as

$$\mathcal{D}_{[\mu} \check{A}^M_{\nu]} = \frac{1}{2} \check{\mathcal{F}}^M_{\mu\nu}, \tag{6.30}$$

and the shorthand notation (cf. eq. (6.10)),

$$D_{MN} \equiv -2\sqrt{3}\Omega_{MN}, \tag{6.31}$$

$J^{\mu\nu}_M$ can be rewritten as

$$J^{\mu\nu}_M = -\frac{i}{2} \Big(\mathcal{N}_{IM} \check{\mathcal{F}}^{\mu\nu I} + \mathcal{N}_{I0} F^{\mu\nu 0} \Big)\Big|_{A^M=0} + \frac{e^{-1}}{4} \epsilon^{\mu\nu\rho\sigma} \Omega_{MN} \check{\mathcal{F}}^N_{\rho\sigma}. \tag{6.32}$$

Inserting this in (6.25) and regrouping some terms, one obtains

$$\begin{aligned} e^{-1}\mathcal{L}^{(4)\mathrm{kin}}_{\mathrm{vec}} = \frac{1}{2}\mathrm{Im}\Big[& \Big(\mathcal{N}_{00} - \mathcal{N}_{0M}\mathcal{N}^{MN}\mathcal{N}_{N0}\Big) F^{0+}_{\mu\nu} F^{\mu\nu 0+} + 2\Big(\mathcal{N}_{0I} - \mathcal{N}_{IM}\mathcal{N}^{MN}\mathcal{N}_{N0}\Big) \check{\mathcal{F}}^{I+}_{\mu\nu} F^{\mu\nu 0+} \\ & + \Big(\mathcal{N}_{IJ} - \mathcal{N}_{IM}\mathcal{N}^{MN}\mathcal{N}_{NJ}\Big) \check{\mathcal{F}}^{I+}_{\mu\nu} \check{\mathcal{F}}^{\mu\nu J+} - 2\Big(D_{PM}\mathcal{N}^{MN}\mathcal{N}_{N0}\Big) \check{\mathcal{F}}^{P+}_{\mu\nu} F^{0+}_{\mu\nu} \\ & + 2\Big(\mathcal{N}_{IM}\mathcal{N}^{MN} D_{NP}\Big) \check{\mathcal{F}}^{I+}_{\mu\nu} \check{\mathcal{F}}^{\mu\nu P+} + \Big(D_{PM}\mathcal{N}^{MN} D_{NQ}\Big) \check{\mathcal{F}}^{P+}_{\mu\nu} \check{\mathcal{F}}^{\mu\nu Q+} \\ & - 2F^{0+}_{\mu\nu} Z^{\mu\nu +} \Big]\Big|_{A^M=0} \end{aligned} \tag{6.33}$$

Eq. (6.33) is now our final form of the dimensionally reduced theory with tensor fields. We will now show that it is "dual" (modulo some field redefinitions) to a standard 4D gauged supergravity theory with the gauge group $K \ltimes \mathcal{H}^{n_T+1}$. Gauging this group in the standard way requires working in the symplectic basis $(\check{X}^A, \check{F}_B)$ and $(\check{F}^A_{\mu\nu}, \check{G}_{\mu\nu B})$, as we discussed at length at the beginning of section 6. As we have seen in section 6.1, the scalars A^M can be "eaten" by the vector fields $\check{A}^M_\mu$ that gauge the translations of $\mathcal{H}^{n_T+1}$. Assuming these scalars to be gauged away from now on, the kinetic term of the $K \ltimes \mathcal{H}^{n_T+1}$ gauged theory is given by

$$\begin{aligned} e^{-1}\check{\mathcal{L}}^{(4)\mathrm{vec}}_{\mathrm{kin}} &= \frac{1}{2}\mathrm{Im}\Big[\check{N}_{AB}\check{\mathcal{F}}^{A+}_{\mu\nu}\check{\mathcal{F}}^{\mu\nu B+}\Big]\Big|_{A^M=0} \\ &= \frac{1}{2}\mathrm{Im}\Big[\check{N}_{00}\check{\mathcal{F}}^{0+}_{\mu\nu}\check{\mathcal{F}}^{\mu\nu 0+} + 2\check{N}_{0I}\check{\mathcal{F}}^{I+}_{\mu\nu}\check{\mathcal{F}}^{\mu\nu 0+} \\ &\qquad + \check{N}_{IJ}\check{\mathcal{F}}^{I+}_{\mu\nu}\check{\mathcal{F}}^{\mu\nu J+} + 2\check{N}_{M0}\check{\mathcal{F}}^{M+}_{\mu\nu}\check{\mathcal{F}}^{0+}_{\mu\nu} \end{aligned}$$

$$+2\check{\mathcal{N}}_{IM}\check{\mathcal{F}}^{I+}_{\mu\nu}\check{\mathcal{F}}^{\mu\nu M+} + \check{\mathcal{N}}_{MN}\check{\mathcal{F}}^{M+}_{\mu\nu}\check{\mathcal{F}}^{\mu\nu N+}\Big]\Big|_{A^M=0}, \tag{6.34}$$

where $\check{\mathcal{N}}_{AB}$ is the period matrix in the basis $(\check{X}^A, \check{F}_B)$ (to be worked out below), and

$$\check{\mathcal{F}}^C_{\mu\nu} = 2\partial_{[\mu}\check{A}^C_{\nu]} + g f^C_{AB}\check{A}^A_\mu \check{A}^B_\nu . \tag{6.35}$$

with the structure constants of eqs. (6.14). Note that, due to $f^B_{0A} = 0$, the vector field $\check{A}^0_\mu$ only appears via its curl in $\check{\mathcal{F}}^0_{\mu\nu}$:

$$\check{\mathcal{F}}^0_{\mu\nu} = 2\partial_{[\mu}\check{A}^0_{\nu]} + g f^0_{MN}\check{A}^M_\mu \check{A}^N_\nu . \tag{6.36}$$

Obviously, (6.33) and (6.34) are not yet of the same form. In fact, there are two important differences:

1. Eq. (6.33) is expressed in terms of the period matrix $\mathcal{N}_{AB}$ that corresponds to the symplectic basis (X^A, F_B). Eq. (6.34), on the other hand, is expressed in terms of the period matrix $\check{\mathcal{N}}_{AB}$ that corresponds to the symplectic section $(\check{X}^A, \check{F}_B)$.

2. Both (6.33) and (6.34) are already expressed in terms of $\check{A}^I_\mu$ and $\check{A}^M_\mu$. However, (6.33) is still expressed in terms of A^0_μ, whereas (6.34) already contains the dual vector field $\check{A}^0_\mu$. Furthermore, (6.33) contains a curious term proportional to (the last term in (6.33))
$$\frac{e^{-1}g}{4}\epsilon^{\mu\nu\rho\sigma}F^0_{\mu\nu}\check{A}^M_\rho \check{A}^N_\sigma . \tag{6.37}$$
Such terms have been analyzed in the literature before [11, 41] (see also the more recent paper [42]). In our case, this term corresponds to some of the standard gauged supergravity terms in (6.34) upon the dualization of $\check{F}^0_{\mu\nu} \leftrightarrow F^0_{\mu\nu}$, as we will show in a moment.

We will now show the equivalence of (6.33) and (6.34) by transforming (6.34) into (6.33). As we have already mentioned, $\check{A}^0_\mu$ appears in (6.34) only via its (Abelian) curl $\check{F}^0_{\mu\nu}$ as it gauges the central charge. In (6.34), $\check{A}^0_\mu$ can therefore be dualized to another vector field C_μ with Abelian field strength $C_{\mu\nu}$. As usual, this is done by adding

$$-\frac{e^{-1}}{4}\epsilon^{\mu\nu\rho\sigma}\check{F}^0_{\mu\nu}C_{\rho\sigma} = \mathrm{Im}[\check{F}^{0+}_{\mu\nu}C^{\mu\nu+}] \tag{6.38}$$

to the Lagrangian (6.34). Varying with respect to $C^+_{\mu\nu}$ and reinserting the resulting equation for $\check{F}^{0+}_{\mu\nu}$ gives

$$e^{-1}\check{\mathcal{L}}^{\text{vec}}_{\text{kin, dual}} = \frac{1}{2}\mathrm{Im}\Big[-2C^+_{\mu\nu}Z^{\mu\nu+} + \Big(\check{\mathcal{N}}_{\tilde{I}\tilde{J}} - \frac{\check{\mathcal{N}}_{0\tilde{I}}\check{\mathcal{N}}_{0\tilde{J}}}{\check{\mathcal{N}}_{00}}\Big)\check{\mathcal{F}}^{\tilde{I}+}_{\mu\nu}\check{\mathcal{F}}^{\mu\nu\tilde{J}+}$$
$$-2\frac{\check{\mathcal{N}}_{0\tilde{I}}}{\check{\mathcal{N}}_{00}}\check{\mathcal{F}}^{\tilde{I}+}_{\mu\nu}C^{\mu\nu+} - \frac{1}{\check{\mathcal{N}}_{00}}C^+_{\mu\nu}C^{\mu\nu+}\Big]\Big|_{A^M=0}. \tag{6.39}$$

In order to bring this to the form (6.33), it remains to reexpress the $\check{\mathcal{N}}_{AB}$ in terms of the $\mathcal{N}_{AB}$ that appear in (6.33). To this end, recall that the basis $(\check{X}^A, \check{F}_B)$ is essentially obtained from the basis (X^A, F_B) by exchange of X^0 with F_0 and X^M with F_M (in fact with $D^{MN}F_N$). This exchange is implemented by the symplectic transformation matrix $\mathcal{S}$ of eq. (6.9):

$$\begin{pmatrix} \check{X}^A \\ \check{F}_B \end{pmatrix} = \mathcal{S} \begin{pmatrix} X^A \\ F_B \end{pmatrix} . \tag{6.40}$$

It is convenient to decompose this transformation into two steps. In the first step, X^M and $D^{MN}F_N$ are exchanged by multiplication with the symplectic matrix

$$\mathcal{S}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta_I^J & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & D^{MN} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta_I^J & 0 \\ 0 & 0 & D_{MN} & 0 & 0 & 0 \end{pmatrix} . \tag{6.41}$$

We call the resulting symplectic vector $(\tilde{X}^A, \tilde{F}_B)$, i.e.

$$\begin{pmatrix} \tilde{X}^A \\ \tilde{F}_B \end{pmatrix} = \mathcal{S}_1 \begin{pmatrix} X^A \\ F_B \end{pmatrix} . \tag{6.42}$$

In a second step, X^0 and F_0 (which are now called $\tilde{X}^0$ and $\tilde{F}_0$) are rotated by subsequent multiplication with the symplectic matrix

$$\mathcal{S}_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \delta_I^J & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta_M^N & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta_I^J & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_M^N \end{pmatrix} . \tag{6.43}$$

Obviously,

$$\mathcal{S} = \mathcal{S}_2 \mathcal{S}_1 \tag{6.44}$$

and

$$\begin{pmatrix} \check{X}^A \\ \check{F}_B \end{pmatrix} = \mathcal{S}_2 \begin{pmatrix} \tilde{X}^A \\ \tilde{F}_B \end{pmatrix} . \tag{6.45}$$

The period matrix is likewise computed in a two step process. First, following eq. (4.26), we determine

$$\tilde{\mathcal{N}} = (C_1 + D_1 \mathcal{N})(A_1 + B_1 \mathcal{N})^{-1} \tag{6.46}$$

where

$$\mathcal{S}_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} . \tag{6.47}$$

The result is

$$\tilde{\mathcal{N}}_{AB} = \begin{pmatrix} \mathcal{N}_{00} - \mathcal{N}_{0M}\mathcal{N}^{MN}\mathcal{N}_{N0} & \mathcal{N}_{0I} - \mathcal{N}_{0M}\mathcal{N}^{MN}\mathcal{N}_{NI} & \mathcal{N}_{0M}\mathcal{N}^{MN}D_{NP} \\ \mathcal{N}_{I0} - \mathcal{N}_{IM}\mathcal{N}^{MN}\mathcal{N}_{N0} & \mathcal{N}_{IJ} - \mathcal{N}_{IM}\mathcal{N}^{MN}\mathcal{N}_{NJ} & \mathcal{N}_{IM}\mathcal{N}^{MN}D_{NP} \\ -D_{MN}\mathcal{N}^{NP}\mathcal{N}_{P0} & -D_{MN}\mathcal{N}^{NP}\mathcal{N}_{PI} & C_{MN}\mathcal{N}^{NP}D_{PR} \end{pmatrix} \tag{6.48}$$

$\check{\mathcal{N}}_{AB}$ can then be obtained from

$$\check{\mathcal{N}} = (C_2 + D_2\tilde{\mathcal{N}})(A_2 + B_2\tilde{\mathcal{N}})^{-1}, \tag{6.49}$$

where

$$\mathcal{S}_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}. \tag{6.50}$$

The result is

$$\check{\mathcal{N}}_{AB} = \frac{1}{\tilde{\mathcal{N}}_{00}} \begin{pmatrix} -1 & \tilde{\mathcal{N}}_{0\tilde{I}} \\ \tilde{\mathcal{N}}_{\tilde{I}0} & (\tilde{\mathcal{N}}_{00}\tilde{\mathcal{N}}_{\tilde{I}\tilde{J}} - \tilde{\mathcal{N}}_{0\tilde{I}}\tilde{\mathcal{N}}_{0\tilde{J}}) \end{pmatrix}. \tag{6.51}$$

We are now ready to show the equivalence of (6.33) and (6.39). First, using (6.51), one rewrites (6.39) as

$$\begin{aligned} e^{-1}\check{\mathcal{L}}^{\text{vec}}_{\text{kin, dual}} = \frac{1}{2}\text{Im}\Big[& -2C^{+}_{\mu\nu}Z^{\mu\nu+} + \tilde{\mathcal{N}}_{\tilde{I}\tilde{J}}\check{\mathcal{F}}^{\tilde{I}+}_{\mu\nu}\check{\mathcal{F}}^{\mu\nu\tilde{J}+} \\ & +2\tilde{\mathcal{N}}_{0\tilde{I}}\check{\mathcal{F}}^{\tilde{I}+}_{\mu\nu}C^{\mu\nu+} + \tilde{\mathcal{N}}_{00}C^{+}_{\mu\nu}C^{\mu\nu+}\Big]\Big|_{A^M=0}. \end{aligned} \tag{6.52}$$

Using (6.48), and identifying

$$F^{0}_{\mu\nu} = C_{\mu\nu}, \tag{6.53}$$

this becomes eq. (6.33).

What we have thus shown, is that after the tensor fields are eliminated, the theory is dual to a standard gauged supergravity theory with gauge group $K \ltimes \mathcal{H}^{n_T+1}$. In order to gauge this group in the standard way, its action has to be made block diagonal on the symplectic section prior to the gauging. This is done by going to a new symplectic basis $(\check{X}^A, \check{F}_B)$, which is obtained from the "natural" basis (X^A, F_B) by exchanging X^0 with F_0 and X^M with $D^{MN}F_N$ by means of a symplectic rotation. The same rotations have to be applied to the corresponding field strengths $(F^0_{\mu\nu}, G_{\mu\nu 0})$ and $(F^M_{\mu\nu}, G_{\mu\nu N})$, where they correspond to electromagnetic duality transformations. After this transformation, the gauging can be carried out in the standard way. In order to recover the compactified theory with the tensor fields eliminated, one finally has to re-dualize $\check{F}^0_{\mu\nu}$ *after* the gauging. This dualization essentially takes back the exchange of X^0 with F_0 (and the corresponding exchange of $F^0_{\mu\nu}$ and $G_{\mu\nu 0}$), but leaves some unusual new couplings of the form (6.37). The vector fields B^M_μ that descend from the 5D tensor fields are interpreted as massive vector fields that gained their mass from eating the scalars A^M, which disappeared from the action. The B^M_μ are essentially the magnetic duals of the A^M_μ of the ungauged theory. This makes sense, as the 5D tensors $\hat{B}^M_{\hat{\mu}\hat{\nu}}$ from which the B^M_μ descend, are also the duals of the 5D vector fields $\hat{A}^M_{\hat{\mu}}$, from which the A^M_μ descend.

6.3 The case of not completely reducible representations

In this subsection, we briefly comment on the dimensionally reduced theory corresponding to case (ii) of section 3, where the decomposition of the $(\tilde{n}+1)$-dimensional representation of G with respect to the the prospective 5D gauge group K is reducible, but not *completely* reducible. This case has been studied in ref. [29]. In that case, the vector fields $\hat{A}^I_{\hat{\mu}}$ still transform in the adjoint representation of $K \subset G$ and have the standard field strengths $\hat{\mathcal{F}}^I_{\hat{\mu}\hat{\nu}} \equiv 2\partial_{[\hat{\mu}}\hat{A}^I_{\hat{\nu}]} + g f^I_{JK}\hat{A}^J_{\hat{\mu}}\hat{A}^K_{\hat{\nu}}$. In addition to the transformation matrix Λ^N_{IM} that acts only on the tensor fields $\hat{B}^M_{\hat{\mu}\hat{\nu}}$, however, there is now also a transformation matrix of the type[9] Λ^M_{IJ} that can mix the tensor fields with the field strengths of the vector fields, so that the representation of K is no longer block diagonal, i.e. completely reducible.

This new matrix is related to a new allowed set of components of the $C_{\tilde{I}\tilde{J}\tilde{K}}$ tensor, namely the components of the form C_{IJM} (which have to vanish in the completely reducible case (i) of section 3):

$$C_{IJM} = -\sqrt{6}\Lambda^N_{(IJ)}\Omega_{NM} \ . \tag{6.54}$$

The modifications that are necessary to perform such a gauging in a supersymmetric way are the same as in the completely reducible case, except for the following differences:

- The covariant derivative (3.6) of the tensor fields, $\hat{\mathcal{D}}_{[\hat{\mu}}\hat{B}^M_{\hat{\nu}\hat{\rho}]}$, in the $\hat{B}^N \wedge \hat{\mathcal{D}}\hat{B}^M$ term of the five-dimensional Lagrangian (3.9) gets an additional contribution due to the mixing matrix Λ^M_{IJ}:

$$\hat{\mathcal{D}}_{[\hat{\mu}}\hat{B}^M_{\hat{\nu}\hat{\rho}]} \to \partial_{[\hat{\mu}}\hat{B}^M_{\hat{\nu}\hat{\rho}]} + 2g\Lambda^M_{IJ}\hat{A}^I_{[\hat{\mu}}\hat{\mathcal{F}}^J_{\hat{\nu}\hat{\rho}]} + g\hat{A}^I_{[\hat{\mu}}\Lambda^M_{IN}\hat{B}^N_{\hat{\nu}\hat{\rho}]}. \tag{6.55}$$

- There are new Chern-Simons terms of the type $AAAF$ and $AAAAA$ beyond the already existing ones that are already displayed in (3.9) for the completely reducible case:

$$\mathcal{L}^{\text{additional}}_{\text{C.-S.}} = -\frac{1}{2}\epsilon^{\hat{\mu}\hat{\nu}\hat{\lambda}\hat{\rho}\hat{\sigma}}\Omega_{MN}\Lambda^M_{IK}\Lambda^N_{FG}\hat{A}^I_{\hat{\mu}}\hat{A}^F_{\hat{\nu}}\hat{A}^G_{\hat{\lambda}}\left(-\frac{1}{2}g\hat{\mathcal{F}}^K_{\hat{\rho}\hat{\sigma}} + \frac{1}{10}g^2 f^K_{HL}\hat{A}^H_{\hat{\rho}}\hat{A}^L_{\hat{\sigma}}\right). \tag{6.56}$$

- The new couplings enter the Killing vectors (and hence the covariant derivatives of the scalars) and the scalar potential via an implicit dependence on Λ^M_{IJ}.

These modifications all have their counterparts in the dimensionally reduced theory in four dimensions, and it is straightforward to determine them from an obvious generalization of the equations displayed in the appendix. One important aspect of the 4D theory, however, can best be seen from the way the non-vanishing C_{IJM} terms influence the transformation laws of the symplectic section under the translation of the Kaluza-Klein scalars A^M by b^M. Indeed, the F_A components of the symplectic section now have additional contributions from the new C_{IJM} terms in the prepotential F, so we now have under

[9]These matrices are called $t_{IJ}{}^M$ in [29].

infinitesimal translation $z^M \to z^M + b^M$,

$$\begin{pmatrix} X^0 \\ X^I \\ X^M \\ F_0 \\ F_I \\ F_M \end{pmatrix} \to \begin{pmatrix} X^0 \\ X^I \\ X^M \\ F_0 \\ F_I \\ F_M \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ b^M X^0 \\ -b^M F_M \\ -2\sqrt{2} b^M C_{MNI} X^N - 2\sqrt{2} b^M C_{IJM} X^J \\ -2\sqrt{2} b^M C_{MIN} X^I \end{pmatrix} . \tag{6.57}$$

We observe that, just as for the completely reducible case, the components (F_0, X^I, F_M) still transform among themselves. However, this is no longer true for the components (X^0, F_I, X^M) if $C_{IJM} \neq 0$ — a clear difference to the completely reducible case with $C_{IJM} = 0$. In fact, the minimal set of components that contains the F_I and closes under translations is in general (X^0, F_I, X^M, X^I), which is too big for one half of a symplectic section. One might wonder whether perhaps some linear combination of the X^M and X^I could be used instead of all the X^M and X^I in this set, so as to make the number of independent components smaller, but this would require a symplectic rotation that somehow trades the F_M with that linear combination of the X^M and X^I, just as we traded F_M and X^M using the matrix $\mathcal{S}_1$ in the completely reducible case. However, whereas $\mathcal{S}_1$ contained only Kronecker deltas and the matrix $D_{MN} \sim \Omega_{MN}$, i.e. a natural object of the 5D theory, there is no natural object with the index structure $\{\cdot\}^I_M$ that one can build from the 5D objects $\Omega_{MN}, f^K_{IJ}, \Lambda^N_{IM}, \Lambda^M_{IJ}, C_{IJK}$, which determine the whole theory. Thus, due to the presence of C_{IJM} terms, it seems in general not possible to find a symplectic matrix $\mathcal{S}$ that brings the gauge transformations to block diagonal form. As a result, these theories in 4D should, apart from some possible special cases, involve topological terms of the form studied in [11, 41] in addition to the standard Yang-Mills gauging, even after dualizations of the type discussed in the previous subsections are performed.

7. CSO$^*(2N)$ gauged supergravity theories from reduction of 5D theories and unified YMESGTs in four dimensions

Unified 5D MESGTs are defined as those theories whose Lagrangian admit a simple symmetry group under which all the vector fields, including the "graviphoton", transform irreducibly. Among those 5D MESGTs whose scalar manifolds are homogeneous spaces only four are unified [43]. They are defined by the four simple Euclidean Jordan algebras of degree three, $J_3^{\mathbb{A}}$, of (3×3) Hermitian matrices over the four division algebras $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ [14], and their scalar manifolds are actually symmetric spaces, which we list below:

$$\begin{aligned} \mathcal{M} &= \mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(3) \qquad (\tilde{n} = 5) \\ \mathcal{M} &= \mathrm{SL}(3,\mathbb{C})/\,\mathrm{SU}(3) \qquad (\tilde{n} = 8) \\ \mathcal{M} &= \mathrm{SU}^*(6)/\,\mathrm{USp}(6) \qquad (\tilde{n} = 14) \\ \mathcal{M} &= E_{6(-26)}/F_4 \qquad\qquad (\tilde{n} = 26), \end{aligned} \tag{7.1}$$

where we have indicated the number of vector multiplets, $\tilde{n}$, for each of these theories. In these cases, the symmetry groups G of these theories are simply the isometry groups $\mathrm{SL}(3,\mathbb{R})$, $\mathrm{SL}(3,\mathbb{C})$, $SU^*(6)$ and $E_{6(-26)}$, respectively, under which the, respectively, 6, 9, 15 and 27 vector fields $A_\mu^{\tilde{I}}$ transform irreducibly [14]. Thus, according to our definition, all of these four theories are unified MESGTs. These supergravity theories are referred to as the magical supergravity theories [44] , because of their deep connection with the Magic Square of Freudenthal, Rozenfeld and Tits [45]. Of these four unified MESGTs in five dimensions only the theory defined by $J_3^{\mathbb{H}}$ can be gauged so as to obtain a unified YMESGT[10] with the gauge group $\mathrm{SO}^*(6) \simeq \mathrm{SU}(3,1)$.

As was shown in [43], if one relaxes the requirement that the scalar manifolds be homogeneous spaces, one finds three infinite families of unified MESGTs in five dimensions. They are defined by Lorentzian Jordan algebras of arbitrary degree over the four associative division algebras $\mathbb{R},\mathbb{C},\mathbb{H}$. These Lorentzian Jordan algebras $J_{(1,N)}^{\mathbb{A}}$ of degree $p = N+1$ are realized by $(N+1)\times(N+1)$ matrices over $\mathbb{A}$ which are hermitian with respect to the Lorentzian metric $\eta = (-,+,+,\ldots,+)$:

$$(\eta X)^\dagger = \eta X \qquad \forall X \in J_{(1,N)}^{\mathbb{A}} \ . \tag{7.2}$$

A general element, U, of $J_{(1,N)}^{\mathbb{A}}$ can be written in the form

$$U = \begin{pmatrix} x & -Y^\dagger \\ Y & Z \end{pmatrix}, \tag{7.3}$$

where Z is an element of the Euclidean subalgebra $J_N^{\mathbb{A}}$ (i.e., it is a Hermitian $(N\times N)$-matrix over $\mathbb{A}$), $x \in \mathbb{R}$, and Y denotes an N-dimensional column vector over $\mathbb{A}$. Under their (non-compact) automorphism group, $\mathrm{Aut}(J_{(1,N)}^{\mathbb{A}})$, these simple Jordan algebras $J_{(1,N)}^{\mathbb{A}}$ decompose into an irreducible representation formed by the traceless elements plus a singlet, which is given by the identity element of $J_{(1,N)}^{\mathbb{A}}$ (i.e., by the unit matrix $\mathbf{1}$):

$$J_{(1,N)}^{\mathbb{A}} = \mathbf{1} \oplus \{\text{traceless elements}\} \equiv \mathbf{1} \oplus J_{(1,N)_0}^{\mathbb{A}}. \tag{7.4}$$

By identifying the structure constants (d-symbols) of the traceless elements of a Lorentzian Jordan algebra $J_{(1,N)}^{\mathbb{A}}$ with the $C_{\tilde{I}\tilde{J}\tilde{K}}$ of a MESGT: $C_{\tilde{I}\tilde{J}\tilde{K}} = d_{\tilde{I}\tilde{J}\tilde{K}}$, one obtains a unified MESGT, in which all the vector fields transform irreducibly under the simple automorphism group $\mathrm{Aut}(J_{(1,N)}^{\mathbb{A}})$ of that Jordan algebra. For $\mathbb{A} = \mathbb{R},\mathbb{C},\mathbb{H}$ one obtains in this way three infinite families of physically acceptable unified MESGTs (one for each $N \geq 2$).

In table 1 below, we list all the simple Lorentzian Jordan algebras of type $J_{(1,N)}^{\mathbb{A}}$, their automorphism groups, and the numbers of vector and scalar fields in the unified 5D MESGTs defined by them.

Note that the number of vector fields for the theories defined by $J_{(1,3)}^{\mathbb{R}}$, $J_{(1,3)}^{\mathbb{C}}$ and $J_{(1,3)}^{\mathbb{H}}$ are 9, 15 and 27, respectively. These are exactly the same numbers of vector fields as in the magical theories based on the norm forms of the Euclidean Jordan algebras $J_3^{\mathbb{C}}$, $J_3^{\mathbb{H}}$

[10] In unified YMESGTs all the vector fields, including the graviphoton, transform in the adjoint representation of a simple gauge group.

J	D	$\mathrm{Aut}(J)$	No. of vector fields	No. of scalars
$J^{\mathbb{R}}_{(1,N)}$	$\frac{1}{2}(N+1)(N+2)$	$\mathrm{SO}(N,1)$	$\frac{1}{2}N(N+3)$	$\frac{1}{2}N(N+3)-1$
$J^{\mathbb{C}}_{(1,N)}$	$(N+1)^2$	$\mathrm{SU}(N,1)$	$N(N+2)$	$N(N+2)-1$
$J^{\mathbb{H}}_{(1,N)}$	$(N+1)(2N+1)$	$\mathrm{USp}(2N,2)$	$N(2N+3)$	$N(2N+3)-1$
$J^{\mathbb{O}}_{(1,2)}$	27	$F_{4(-20)}$	26	25

Table 1: List of the simple Lorentzian Jordan algebras of type $J^{\mathbb{A}}_{(1,N)}$. The columns show, respectively, their dimensions D, their automorphism groups $\mathrm{Aut}(J^{\mathbb{A}}_{(1,N)})$, the number of vector fields $(\tilde{n}+1) = (D-1)$ and the number of scalars $\tilde{n} = (D-2)$ in the corresponding MESGTs.

and $J^{\mathbb{O}}_3$, respectively (cf. eq. (7.1)). As was shown in [43], this is not an accident; the magical MESGTs based on $J^{\mathbb{C}}_3$, $J^{\mathbb{H}}_3$ and $J^{\mathbb{O}}_3$ found in [14] are *equivalent* (i.e. the cubic polynomials $\mathcal{V}(h)$ agree) to the ones defined by the Lorentzian algebras $J^{\mathbb{R}}_{(1,3)}$, $J^{\mathbb{C}}_{(1,3)}$ and $J^{\mathbb{H}}_{(1,3)}$, respectively. This is a consequence of the fact that the generic norms of the degree 3 simple Euclidean Jordan algebras $J^{\mathbb{C}}_3$, $J^{\mathbb{H}}_3$ and $J^{\mathbb{O}}_3$ coincide with the cubic norms defined over the traceless elements of degree four simple Lorentzian Jordan algebras over $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ [43]. This implies that the only known unified MESGT that is not covered by the table 1 is the magical theory of [14] based on the Euclidean Jordan algebra $J^{\mathbb{R}}_3$ with $(\tilde{n}+1) = 6$ vector fields and the target space $\mathcal{M} = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$. Except for the theories defined by $J^{\mathbb{R}}_{(1,3)}$, $J^{\mathbb{C}}_{(1,3)}$ and $J^{\mathbb{H}}_{(1,3)}$ the scalar manifolds of MESGTs defined by other simple Lorentzian Jordan algebras are not homogeneous.

Of these three infinite families of unified MESGTs in five dimensions only the family defined by $J^{\mathbb{C}}_{(1,N)}$ can be gauged so as to obtain an infinite family of unified YMESGTs with the gauge groups $\mathrm{SU}(N,1)$ [43]. As for the family defined by the quaternionic Lorentzian Jordan algebras $J^{\mathbb{H}}_{(1,N)}$, they can be gauged with the gauge groups $\mathrm{SU}(N,1)$ while dualizing the remaining $N(N+1)$ vector fields to tensor fields in five dimensions.

Let us now analyze the dimensional reduction of the 5D YMESGTs with the gauge group $\mathrm{SU}(N,1)$ coupled to $N(N+1)$ tensor fields. From the results of section 6 it follows that the corresponding four dimensional theory is dual to a standard $\mathcal{N}=2$ YMESGT with the gauge group $\mathrm{SU}(N,1) \ltimes \mathcal{H}^{N(N+1)+1}$. However, the group $\mathrm{SU}(N,1) \ltimes \mathcal{H}^{N(N+1)+1}$ can be obtained by contraction from the simple noncompact group $\mathrm{SO}^*(2N+2)$. This is best seen by considering the three graded decomposition of the Lie algebra of $\mathrm{SO}^*(2N+2)$ with respect to the Lie algebra of its subgroup $SU(N,1) \times \mathrm{U}(1)$

$$\mathfrak{so}^*(2N+2) = \mathfrak{g}^{-1} \oplus [\mathfrak{su}(N,1) \times \mathfrak{u}(1)] \oplus \mathfrak{g}^{+1}$$

where grade $+1$ and -1 subspaces transform in the antisymmetric tensor representation of $\mathrm{SU}(N,1)$ and its conjugate, respectively. By rescaling the generators belonging to the grade ± 1 spaces and redefining the generators in the limit in which the scale parameter goes to infinity one obtains the Lie algebra isomorphic to the Lie algebra of $\mathrm{SU}(N,1) \ltimes \mathcal{H}^{N(N+1)+1}$. Such contractions arise in the pp-wave limits of spacetime groups and were studied in [46]. We shall denote the contracted algebra as $\mathrm{CSO}^*[2N+2\|\,\mathrm{U}(N,1)]$. For $N=3$, $\mathrm{CSO}^*[8\|\,\mathrm{U}(3,1)]$ coincides with the contraction of $\mathrm{SO}^*(8)$ denoted as $\mathrm{CSO}^*(6,2)$ by Hull [23] since $\mathrm{SO}^*(6)$ is isomorphic to $\mathrm{SU}(3,1)$.

Now the MESGT theory defined by $J^{\mathbb{H}}_{(1,3)}$ can be gauged directly in four dimensions so as to obtain a unified YMESGT with the gauge group $\mathrm{SO}^*(8)=\mathrm{SO}(6,2)$. By contracting this unified theory, one can obtain the $\mathrm{CSO}^*[8\|\,\mathrm{U}(3,1)]=\mathrm{CSO}^*(6,2)$ gauging directly in four dimensions, which is consistent with the above observation.

In [31] it was pointed out that the three infinite families of 4D MESGTs defined by Lorentzian Jordan algebras might admit symplectic sections in which all the vector fields transform irreducibly under the reduced structure groups of the corresponding Jordan algebras. Since their reduced structure groups are simple they would be unified MESGTs in four dimensions as well. Of these three infinite families of unified MESGTs only the family defined by the quaternionic Jordan algebras $J^{\mathbb{H}}_{(1,N)}$ could then be gauged so as to obtain unified YMESGTs with gauge groups $\mathrm{SO}^*(2N+2)$ in four dimensions. The fact that the dimensional reduction of the five dimensional gauged YMESGTs with gauge groups $\mathrm{SU}(N,1)$ coupled to $N(N+1)$ tensor fields leads to contracted versions of the $\mathrm{SO}^*(2N+2)$ gauged YMESGTs is evidence for the existence of this infinite family of unified YMESGTs.

8. Some comments on the scalar potential

We already mentioned in the Introduction that 5D noncompact YMESGTs with tensor multiplets and R-symmetry gauging provide the only known examples of stable de Sitter ground states in higher-dimensional gauged supergravity theories [24, 27]. In this paper, we considered the dimensional reduction of 5D YMESGTs with tensor fields (but without R-symmetry gauging) to 4D and found that the resulting theories have non-Abelian non-compact gauge groups in 4D which are of the form $K\ltimes\mathcal{H}^{n_T+1}$. Non-compact non-Abelian gauge groups were also found essential for stable de Sitter vacua in 4D, $\mathcal{N}=2$ supergravity in [26]. Interestingly, the vectors that gauge the Heisenberg algebra require a symplectic rotation relative to the vector fields that gauge the 5D part K of the 4D gauge group in order to bring the 4D gauging into the standard block diagonal form. Apart from the semi-direct vs. direct structure, this is reminiscent of the de Roo-Wagemans angles that were also found to be important for stable de Sitter ground states in 4D, $\mathcal{N}=2$ supergravity in [26]. As a third ingredient for stable de Sitter vacua in 4D, $\mathcal{N}=2$ supergravity, the authors of [26] identified gaugings of the R-symmetry group, which are also important in 5D [24, 27].

One might now wonder whether these findings might perhaps have something to with each other. Let us therefore take a look at the scalar potential of the dimensional reduced YMESGTs with tensor fields. From eq. (4.37), we have

$$P=e^{-\sigma}P^{(T)}(h^{\tilde{I}})+\frac{3}{4}e^{-3\sigma}\overset{\circ}{a}_{\tilde{I}\tilde{J}}(A^IM^{\tilde{I}}_{I\tilde{K}}h^{\tilde{K}})(A^JM^{\tilde{J}}_{J\tilde{L}}h^{\tilde{L}}) \tag{8.1}$$

where the first term is simply the dimensional reduction of the 5D scalar potential and the second term comes from the 5D kinetic term of the scalar fields. If we had instead started from a 5D, $\mathcal{N} = 2$ YMESGT with tensor fields *and* R-symmetry gauging, we would have gotten an additional term in 4D of the form

$$e^{-\sigma} P^{(R)}(h^{\tilde{I}}) \tag{8.2}$$

which is just the dimensional reduction of the 5D scalar potential due to the R-symmetry gauging (the R-symmetry gauging does not affect the scalar fields, and therefore there is no analogue of the second term of eq. (8.1) in addition to the already existing one.). It is easy to convince oneself that the second term in (8.1) is a positive definite real form for the A^I. A solution with $\langle A^I \rangle = 0$ is therefore a solution without tachyonic directions in the A^I space. The first term in (8.1) and the term (8.2) only depend on the $h^{\tilde{I}}$ and σ. Setting the $h^{\tilde{I}}$ equal to their values that are known to lead to a stable de Sitter vacuum in five dimensions in the models discussed in [24, 27] would then lead to a de Sitter point in 4D as well with the $h^{\tilde{I}}$ having positive masses. Unfortunately, however, this point would not be a critical point of the potential due to the runaway behaviour in the σ direction. In order to fix σ at finite values, one would have to allow the second term in (8.1) to be non-zero. But then one would have to be at a point where $\langle A^I \rangle \neq 0$, which might require other values of the $h^{\tilde{I}}$ that no longer correspond to the stable de Sitter vacua that are known from five dimensions.

A careful investigation of the scalar potential (8.1) perhaps together with a gauging of the 4D R-symmetry group might lead to many interesting types of critical points, but is beyond the scope of this paper.

A. Details of the dimensional reduction

This appendix lists the the dimensional reductions of the individual terms of the Lagrangian (3.9) using the decompositions (4.1), (4.3) and (4.32).

A.1 The Einstein-Hilbert term

The Einstein-Hilbert term in (3.9) leads to the same four-dimensional terms as in the ungauged case,

$$\begin{aligned} \mathcal{L}^{(5)}_{\text{E.-H.}} &\equiv -\frac{1}{2}\hat{e}\hat{R} \Rightarrow \\ e^{-1}\mathcal{L}^{(4)}_{\text{E.-H.}} &= -\frac{1}{2}R - \frac{1}{2}e^{3\sigma}W_{\mu\nu}W^{\mu\nu} - \frac{3}{4}\partial_\mu\sigma\partial^\mu\sigma \end{aligned} \tag{A.1}$$

A.2 The $\hat{\mathcal{H}}\hat{\mathcal{H}}$-term

The $\hat{\mathcal{H}}\hat{\mathcal{H}}$-term in (3.9) reduces to

$$\begin{aligned} \mathcal{L}^{(5)}_{\hat{\mathcal{H}}\hat{\mathcal{H}}} &\equiv -\frac{1}{4}\hat{e}\mathring{a}_{\tilde{I}\tilde{J}}\hat{\mathcal{H}}^{\tilde{I}}_{\hat{\mu}\hat{\nu}}\hat{\mathcal{H}}^{\tilde{J}\hat{\mu}\hat{\nu}} \Rightarrow \\ e^{-1}\mathcal{L}^{(4)}_{\hat{\mathcal{H}}\hat{\mathcal{H}}} &= -\frac{1}{4}e^{\sigma}\mathring{a}_{IJ}(\mathcal{F}^I_{\mu\nu} + 2W_{\mu\nu}A^I)(\mathcal{F}^{J\mu\nu} + 2W^{\mu\nu}A^J) \\ &\quad -\frac{1}{2}e^{\sigma}\mathring{a}_{IM}(\mathcal{F}^I_{\mu\nu} + 2W_{\mu\nu}A^I)B^{M\mu\nu} - \frac{1}{4}e^{\sigma}\mathring{a}_{MN}B^M_{\mu\nu}B^{N\mu\nu} \end{aligned}$$

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$$
\begin{aligned}
&-\frac{1}{2}e^{-2\sigma}\overset{\circ}{a}_{IJ}(\mathcal{D}_\mu A^I)(\mathcal{D}^\mu A^J) - e^{-2\sigma}\overset{\circ}{a}_{IM}(\mathcal{D}_\mu A^I)B^{\mu M} \\
&-\frac{1}{2}e^{-2\sigma}\overset{\circ}{a}_{MN}B^M_\mu B^{\mu N},
\end{aligned} \tag{A.2}
$$

where

$$
\mathcal{D}_\mu A^I \equiv \partial_\mu A^I + g A^J_\mu f^I_{JK} A^K \tag{A.3}
$$

$$
\mathcal{F}^I_{\mu\nu} \equiv 2\partial_{[\mu} A^I_{\nu]} + g f^I_{JK} A^J_\mu A^K_\nu . \tag{A.4}
$$

A.3 The scalar kinetic term

Using (2.5) and

$$
K^{\tilde{x}}_I(\partial_{\tilde{x}} h^{\tilde{I}}) = M^{\tilde{I}}_{(I)\tilde{J}} h^{\tilde{J}}, \tag{A.5}
$$

the 5D scalar kinetic term can be rewritten as

$$
\mathcal{L}^{(5)}_{\text{scalar}} \equiv -\frac{\hat{e}}{2} g_{\tilde{x}\tilde{y}}(\hat{\mathcal{D}}_{\hat{\mu}}\varphi^{\tilde{x}})(\hat{\mathcal{D}}^{\hat{\mu}}\varphi^{\tilde{y}}) = -\frac{3\hat{e}}{4}\overset{\circ}{a}_{\tilde{I}\tilde{J}}(\hat{\mathcal{D}}_{\hat{\mu}} h^{\tilde{I}})(\hat{\mathcal{D}}^{\hat{\mu}} h^{\tilde{J}}), \tag{A.6}
$$

where

$$
\hat{\mathcal{D}}_{\hat{\mu}} h^{\tilde{I}} \equiv \partial_{\hat{\mu}} h^{\tilde{I}} + g\hat{A}^I_{\hat{\mu}} M^{\tilde{I}}_{I\tilde{K}} h^{\tilde{K}}. \tag{A.7}
$$

Upon dimensional reduction, this becomes

$$
\begin{aligned}
\mathcal{L}^{(4)}_{\text{scalar}} &= -\frac{3e}{4}\overset{\circ}{a}_{\tilde{I}\tilde{J}}(\mathcal{D}_\mu h^{\tilde{I}})(\mathcal{D}^\mu h^{\tilde{J}}) \\
&\quad -\frac{3e}{4} g^2 e^{-3\sigma}\overset{\circ}{a}_{\tilde{I}\tilde{J}}(A^I M^{\tilde{I}}_{I\tilde{K}} h^{\tilde{K}})(A^J M^{\tilde{J}}_{J\tilde{L}} h^{\tilde{L}}),
\end{aligned} \tag{A.8}
$$

where now the covariant derivative is with respect to the Kaluza-Klein invariant vector fields, A^I_μ,

$$
\mathcal{D}_\mu h^{\tilde{I}} \equiv \partial_\mu h^{\tilde{I}} + g A^I_\mu M^{\tilde{I}}_{I\tilde{K}} h^{\tilde{K}}. \tag{A.9}
$$

A.4 The Chern-Simons term

The 5D Chern-simons term

$$
\begin{aligned}
\hat{e}^{-1}\mathcal{L}^{(5)}_{\text{C.S.}} \equiv \frac{\hat{e}^{-1}}{6\sqrt{6}} C_{IJK}\hat{\epsilon}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\lambda}} \Big\{ & \hat{F}^I_{\hat{\mu}\hat{\nu}}\hat{F}^J_{\hat{\rho}\hat{\sigma}}\hat{A}^K_{\hat{\lambda}} + \frac{3}{2} g \hat{F}^I_{\hat{\mu}\hat{\nu}}\hat{A}^J_{\hat{\rho}}(f^K_{LF}\hat{A}^L_{\hat{\sigma}}\hat{A}^F_{\hat{\lambda}}) \\
& + \frac{3}{5} g^2 (f^J_{GH}\hat{A}^G_{\hat{\nu}}\hat{A}^H_{\hat{\rho}})(f^K_{LF}\hat{A}^L_{\hat{\sigma}}\hat{A}^F_{\hat{\lambda}})\hat{A}^I_{\hat{\mu}} \Big\}
\end{aligned} \tag{A.10}
$$

reduces as follows

$$
\begin{aligned}
e^{-1}\mathcal{L}^{(4)}_{\text{C.S.}} = \frac{e^{-1}}{2\sqrt{6}} C_{IJK}\epsilon^{\mu\nu\rho\sigma} \Big\{ & \mathcal{F}^I_{\mu\nu}\mathcal{F}^J_{\rho\sigma} A^K + 2\mathcal{F}^I_{\mu\nu} W_{\rho\sigma} A^J A^K \\
& + \frac{4}{3} W_{\mu\nu} W_{\rho\sigma} A^I A^J A^K \Big\}
\end{aligned} \tag{A.11}
$$

A.5 The $\hat{B}\hat{\mathcal{D}}\hat{B}$ term

Using the decomposition (4.32), the 5D $\hat{B}\hat{\mathcal{D}}\hat{B}$ term becomes

$$\begin{aligned}
\mathcal{L}^{(5)}_{\hat{B}\hat{\mathcal{D}}\hat{B}} &\equiv \frac{1}{4g}\hat{\epsilon}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\lambda}}\Omega_{MN}\hat{B}^M_{\hat{\mu}\hat{\nu}}\hat{\mathcal{D}}_{\hat{\rho}}\hat{B}^N_{\hat{\sigma}\hat{\lambda}} \\
&= \frac{1}{g}\epsilon^{\mu\nu\rho\sigma}\Omega_{MN}B^M_{\mu\nu}(\partial_\rho B^N_\sigma + gA^I_\rho\Lambda^N_{IP}B^P_\sigma) \\
&\quad + \frac{1}{g}\epsilon^{\mu\nu\rho\sigma}\Omega_{MN}W_{\mu\nu}B^M_\rho B^N_\sigma + \frac{1}{2\sqrt{6}}C_{MNI}\epsilon^{\mu\nu\rho\sigma}B^M_{\mu\nu}B^N_{\rho\sigma}A^I.
\end{aligned} \tag{A.12}$$

A.6 The 5D scalar potential

The 5D scalar potential term reduces to

$$\mathcal{L}^{(5)}_{\mathrm{pot}} \equiv -\hat{e}g^2P^{(T)} = -g^2ee^{-\sigma}P^{(T)}. \tag{A.13}$$

Putting everything together, and regrouping some terms, one then arrives at the dimensionally reduced YMESGT with tensor fields written in eq. (4.33).

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References

[1] B. de Wit, R. Philippe and A. Van Proeyen, *The improved tensor multiplet in $N=2$ supergravity*, *Nucl. Phys.* **B 219** (1983) 143;
N. Berkovits and W. Siegel, *Superspace effective actions for 4d compactifications of heterotic and type-II superstrings*, *Nucl. Phys.* **B 462** (1996) 213 [`hep-th/9510106`];
B. de Wit, M. Roček and S. Vandoren, *Hypermultiplets, hyperkaehler cones and quaternion-Kähler geometry*, *JHEP* **02** (2001) 039 [`hep-th/0101161`].

[2] J. Louis and A. Micu, *Type II theories compactified on Calabi-Yau threefolds in the presence of background fluxes*, *Nucl. Phys.* **B 635** (2002) 395 [`hep-th/0202168`];
U. Theis and S. Vandoren, *$N=2$ supersymmetric scalar-tensor couplings*, *JHEP* **04** (2003) 042 [`hep-th/0303048`];
U. Theis and S. Vandoren, *Instantons in the double-tensor multiplet*, *JHEP* **09** (2002) 059 [`hep-th/0208145`];
M. Davidse, M. de Vroome, U. Theis and S. Vandoren, *Instanton solutions for the universal hypermultiplet*, *Fortschr. Phys.* **52** (2004) 696 [`hep-th/0309220`];

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G. Dall'Agata, R. D'Auria, L. Sommovigo and S. Vaulà, $D = 4$, $N = 2$ *gauged supergravity in the presence of tensor multiplets*, *Nucl. Phys.* **B 682** (2004) 243 [hep-th/0312210];
L. Sommovigo and S. Vaulà, $D = 4$, $N = 2$ *supergravity with abelian electric and magnetic charge*, *Phys. Lett.* **B 602** (2004) 130 [hep-th/0407205];
R. D'Auria, L. Sommovigo and S. Vaulà, $N = 2$ *supergravity lagrangian coupled to tensor multiplets with electric and magnetic fluxes*, *JHEP* **11** (2004) 028 [hep-th/0409097];
R. D'Auria and S. Ferrara, *Dyonic masses from conformal field strengths in d even dimensions*, *Phys. Lett.* **B 606** (2005) 211 [hep-th/0410051];
R. D'Auria, S. Ferrara, M. Trigiante and S. Vaulà, *Scalar potential for the gauged heisenberg algebra and a non-polynomial antisymmetric tensor theory*, *Phys. Lett.* **B 610** (2005) 270 [hep-th/0412063];
L. Sommovigo, *Poincaré dual of* $D = 4$ $N = 2$ *supergravity with tensor multiplets*, *Nucl. Phys.* **B 716** (2005) 248 [hep-th/0501048];
R. D'Auria, S. Ferrara, M. Trigiante and S. Vaulà, $N = 1$ *reductions of* $N = 2$ *supergravity in the presence of tensor multiplets*, *JHEP* **03** (2005) 052 [hep-th/0502219];
G. Dall'Agata, R. D'Auria and S. Ferrara, *Compactifications on twisted tori with fluxes and free differential algebras*, *Phys. Lett.* **B 619** (2005) 149 [hep-th/0503122];
L. Andrianopoli, S. Ferrara, M.A. Lledó and O. Macia, *Integration of massive states as contractions of non linear sigma-models*, *J. Math. Phys.* **46** (2005) 072307 [hep-th/0503196].

[3] J. Louis and W. Schulgin, *Massive tensor multiplets in* $N = 1$ *supersymmetry*, *Fortschr. Phys.* **53** (2005) 235 [hep-th/0410149];
S.M. Kuzenko, *On massive tensor multiplets*, *JHEP* **01** (2005) 041 [hep-th/0412190].

[4] U. Theis, *Masses and dualities in extended Freedman-Townsend models*, *Phys. Lett.* **B 609** (2005) 402 [hep-th/0412177].

[5] C. Bachas, *A way to break supersymmetry*, hep-th/9503030;
J. Polchinski and A. Strominger, *New vacua for type-II string theory*, *Phys. Lett.* **B 388** (1996) 736 [hep-th/9510227];
J. Michelson, *Compactifications of type-IIB strings to four dimensions with non-trivial classical potential*, *Nucl. Phys.* **B 495** (1997) 127 [hep-th/9610151].

[6] J. Scherk and J.H. Schwarz, *How to get masses from extra dimensions*, *Nucl. Phys.* **B 153** (1979) 61.

[7] A. Strominger, *Massless black holes and conifolds in string theory*, *Nucl. Phys.* **B 451** (1995) 96 [hep-th/9504090].

[8] T. Mohaupt and M. Zagermann, *Gauged supergravity and singular Calabi-Yau manifolds*, *JHEP* **12** (2001) 026 [hep-th/0109055].

[9] Y. Takahashi and R. Palmer, *Gauge-independent formulation of a massive field with spin one*, *Phys. Rev.* **D 1** (1970) 2974;
F. Quevedo and C.A. Trugenberger, *Phases of antisymmetric tensor field theories*, *Nucl. Phys.* **B 501** (1997) 143 [hep-th/9604196];
A. Smailagic and E. Spallucci, *The dual phases of massless/massive Kalb-Ramond fields*, *J. Phys.* **A 34** (2001) L435 [hep-th/0106173].

[10] B. de Wit and A. Van Proeyen, *Potentials and symmetries of general gauged* $N = 2$ *supergravity–Yang-Mills models*, *Nucl. Phys.* **B 245** (1984) 89;
J. Bagger and E. Witten, *Matter couplings in* $N = 2$ *supergravity*, *Nucl. Phys.* **B 222** (1983) 1;

R. D'Auria, S. Ferrara and P. Fre, *Special and quaternionic isometries: general couplings in $N = 2$ supergravity and the scalar potential*, *Nucl. Phys.* **B 359** (1991) 705;
L. Andrianopoli et al., *$N = 2$ supergravity and $N = 2$ super Yang-Mills theory on general scalar manifolds: symplectic covariance, gaugings and the momentum map*, *J. Geom. Phys.* **23** (1997) 111 [hep-th/9605032].

[11] B. de Wit, P.G. Lauwers and A. Van Proeyen, *Lagrangians of $N = 2$ supergravity-matter systems*, *Nucl. Phys.* **B 255** (1985) 569.

[12] B. de Wit, H. Samtleben and M. Trigiante, *Magnetic charges in local field theory*, *JHEP* **09** (2005) 016 [hep-th/0507289].

[13] M. Günaydin and M. Zagermann, *The gauging of five-dimensional, $N = 2$ Maxwell-Einstein supergravity theories coupled to tensor multiplets*, *Nucl. Phys.* **B 572** (2000) 131 [hep-th/9912027].

[14] M. Günaydin, G. Sierra and P.K. Townsend, *The geometry of $N = 2$ Maxwell-einstein supergravity and Jordan algebras*, *Nucl. Phys.* **B 242** (1984) 244.

[15] P.K. Townsend, K. Pilch and P. van Nieuwenhuizen, *Selfduality in odd dimensions*, *Phys. Lett.* **B 136** (1984) 38.

[16] M. Günaydin, L.J. Romans and N.P. Warner, *Gauged $N = 8$ supergravity in five-dimensions*, *Phys. Lett.* **B 154** (1985) 268; *Compact and noncompact gauged supergravity theories in five-dimensions*, *Nucl. Phys.* **B 272** (1986) 598.

[17] M. Pernici, K. Pilch and P. van Nieuwenhuizen, *Gauged $N = 8$ $D = 5$ supergravity*, *Nucl. Phys.* **B 259** (1985) 460.

[18] L.J. Romans, *Gauged $N = 4$ supergravities in five-dimensions and their magnetovac backgrounds*, *Nucl. Phys.* **B 267** (1986) 433.

[19] G. Dall'Agata, C. Herrmann and M. Zagermann, *General matter coupled $N = 4$ gauged supergravity in five dimensions*, *Nucl. Phys.* **B 612** (2001) 123 [hep-th/0103106].

[20] B. de Wit and H. Samtleben, *Gauged maximal supergravities and hierarchies of nonabelian vector-tensor systems*, *Fortschr. Phys.* **53** (2005) 442 [hep-th/0501243];
B. de Wit, H. Samtleben and M. Trigiante, *The maximal $D = 5$ supergravities*, *Nucl. Phys.* **B 716** (2005) 215 [hep-th/0412173].

[21] M. Günaydin and N. Marcus, *The spectrum of the S^5 compactification of the chiral $N = 2$, $D = 10$ supergravity and the unitary supermultiplets of* $\mathrm{U}(2, 2/4)$, *Class. and Quant. Grav.* **2** (1985) L11; *The unitary supermultiplet of $N = 8$ conformal superalgebra involving fields of spin ≤ 2*, *Class. and Quant. Grav.* **2** (1985) L19.

[22] M. Günaydin, L.J. Romans and N.P. Warner, *IIB, or not IIB: that is the question*, *Phys. Lett.* **B 164** (1985) 309;
A realistic four-dimensional field theory from the IIB superstring?, CALT-68-1295.

[23] C.M. Hull, *New gauged $N = 8$, $D = 4$ supergravities*, *Class. and Quant. Grav.* **20** (2003) 5407 [hep-th/0204156].

[24] M. Günaydin and M. Zagermann, *The vacua of 5d, $N = 2$ gauged Yang-Mills/Einstein/tensor supergravity: abelian case*, *Phys. Rev.* **D 62** (2000) 044028 [hep-th/0002228].

[25] C.M. Hull, *de Sitter space in supergravity and M-theory*, *JHEP* **11** (2001) 012 [hep-th/0109213]; *Domain wall and de Sitter solutions of gauged supergravity*, *JHEP* **11** (2001) 061 [hep-th/0110048];
G.W. Gibbons and C.M. Hull, *de Sitter space from warped supergravity solutions*, hep-th/0111072;
R. Kallosh, A.D. Linde, S. Prokushkin and M. Shmakova, *Gauged supergravities, de Sitter space and cosmology*, *Phys. Rev.* **D 65** (2002) 105016 [hep-th/0110089];
R. Kallosh, *Supergravity, M-theory and cosmology*, hep-th/0205315;
M. de Roo, D.B. Westra and S. Panda, *de Sitter solutions in $N = 4$ matter coupled supergravity*, *JHEP* **02** (2003) 003 [hep-th/0212216];
M. de Roo, D.B. Westra, S. Panda and M. Trigiante, *Potential and mass-matrix in gauged $N = 4$ supergravity*, *JHEP* **11** (2003) 022 [hep-th/0310187].

[26] P. Fré, M. Trigiante and A. Van Proeyen, *Stable de Sitter vacua from $N = 2$ supergravity*, *Class. and Quant. Grav.* **19** (2002) 4167 [hep-th/0205119];
P. Fré, M. Trigiante and A. Van Proeyen, *$N = 2$ supergravity models with stable de Sitter vacua*, *Class. and Quant. Grav.* **20** (2003) S487 [hep-th/0301024].

[27] B. Cosemans and G. Smet, *Stable de Sitter vacua in $N = 2$, $D = 5$ supergravity*, *Class. and Quant. Grav.* **22** (2005) 2359 [hep-th/0502202].

[28] M. de Roo and P. Wagemans, *Gauge matter coupling in $N = 4$ supergravity*, *Nucl. Phys.* **B 262** (1985) 644.

[29] E. Bergshoeff et al., *$N = 2$ supergravity in five dimensions revisited*, *Class. and Quant. Grav.* **21** (2004) 3015 [hep-th/0403045].

[30] D.Z. Freedman and P.K. Townsend, *Antisymmetric tensor gauge theories and nonlinear sigma models*, *Nucl. Phys.* **B 177** (1981) 282.

[31] M. Günaydin, S. McReynolds and M. Zagermann, *Unified $N = 2$ Maxwell-Einstein and Yang-Mills-einstein supergravity theories in four dimensions*, *JHEP* **09** (2005) 026 [hep-th/0507227].

[32] M. Günaydin, G. Sierra and P.K. Townsend, *Gauging the $D = 5$ Maxwell-einstein supergravity theories: more on Jordan algebras*, *Nucl. Phys.* **B 253** (1985) 573.

[33] G. Sierra, *$N = 2$ Maxwell matter Einstein supergravities in $D = 5$, $D = 4$ and $D = 3$*, *Phys. Lett.* **B 157** (1985) 379.

[34] A. Ceresole and G. Dall'Agata, *General matter coupled $N = 2$, $D = 5$ gauged supergravity*, *Nucl. Phys.* **B 585** (2000) 143 [hep-th/0004111].

[35] M. Günaydin and M. Zagermann, *Gauging the full R-symmetry group in five-dimensional, $N = 2$ Yang-Mills/einstein/tensor supergravity*, *Phys. Rev.* **D 63** (2001) 064023 [hep-th/0004117].

[36] B. de Wit and A. Van Proeyen, *Special geometry, cubic polynomials and homogeneous quaternionic spaces*, *Commun. Math. Phys.* **149** (1992) 307 [hep-th/9112027].

[37] B. de Wit and A. Van Proeyen, *Broken sigma model isometries in very special geometry*, *Phys. Lett.* **B 293** (1992) 94 [hep-th/9207091].

[38] B. Craps, F. Roose, W. Troost and A. Van Proeyen, *Special Kähler geometry: does there exist a prepotential?*, hep-th/9712092.

[39] See , for example, B. de Wit, H. Samtleben and M. Trigiante, *On lagrangians and gaugings of maximal supergravities*, *Nucl. Phys.* **B 655** (2003) 93 [hep-th/0212239], and the references therein.

[40] B. de Wit, F. Vanderseypen and A. Van Proeyen, *Symmetry structure of special geometries*, *Nucl. Phys.* **B 400** (1993) 463 [hep-th/9210068].

[41] B. de Wit, C.M. Hull and M. Roček, *New topological terms in gauge invariant actions*, *Phys. Lett.* **B 184** (1987) 233.

[42] L. Andrianopoli, S. Ferrara and M.A. Lledó, *Axion gauge symmetries and generalized Chern-Simons terms in $N = 1$ supersymmetric theories*, *JHEP* **04** (2004) 005 [hep-th/0402142].

[43] M. Günaydin and M. Zagermann, *Unified Maxwell-Einstein and Yang-Mills-Einstein supergravity theories in five dimensions*, *JHEP* **07** (2003) 023 [hep-th/0304109].

[44] M. Günaydin, G. Sierra and P.K. Townsend, *Exceptional supergravity theories and the magic square*, *Phys. Lett.* **B 133** (1983) 72.

[45] H. Freudenthal, *Nederl. Akad. Wetensch. Proc.* **A62** (1959) 447;
B.A. Rozenfeld, *Dokl. Akad. Nauk. SSSR* **106** (1956) 600;
J. Tits, *Mem. Acad. Roy. Belg. Sci.* **29** (1955) fasc. 3.

[46] S. Fernando, M. Günaydin and S. Hyun, *Oscillator construction of spectra of pp-wave superalgebras in eleven dimensions*, *Nucl. Phys.* **B 727** (2005) 421 [hep-th/0411281];
S. Fernando, M. Günaydin and O. Pavlyk, *Spectra of pp-wave limits of M-/superstring theory on $AdS_p \times S^q$ spaces*, *JHEP* **10** (2002) 007 [hep-th/0207175].

[47] M. Pernici, K. Pilch and P. van Nieuwenhuizen, *Gauged maximally extended supergravity in seven-dimensions*, *Phys. Lett.* **B 143** (1984) 103.